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Macdonald polynomials and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$

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ABSTRACT

We define a family of symmetric polynomials $G_{\nu,\lambda}(z_1,\dots,z_{n+1},q)$ indexed by a pair of dominant integral weights for a root system of type A_n. The polynomial $G_{\nu,0}(z,q)$ is the specialized Macdonald polynomial $P_\nu(z,q,0)$ and is known to be the graded character of a level one Demazure module associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. We prove that $G_{0,\lambda}(z,q)$ is the graded character of a level two Demazure module for $\widehat{\mathfrak{sl}}_{n+1}$. Under suitable conditions on (ν,λ) (which apply to the pairs $(\nu,0)$ and $(0,\lambda)$) we prove that $G_{\nu,\lambda}(z,q)$ is Schur positive, i.e., it can be written as a linear combination of Schur polynomials with coefficients in $\mathbb{Z}_+[q]$. We further prove that $P_\nu(z,q,0)$ is a linear combination of elements $G_{0,\lambda}(z,q)$ with the coefficients being essentially products of q-binomials. Together with a result of K. Naoi, a consequence of our result is an explicit formula for the specialized Macdonald polynomial associated to a non-simply laced Lie algebra as a linear combi-

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nation of the level one Demazure characters of the non-simply laced algebra.

0. Introduction

In 1987, I.G. Macdonald introduced a family of orthogonal symmetric polynomials $P_\lambda(z,q,t)$, $z=(z_1,\dots,z_{n+1})$ which are a basis for the ring of symmetric polynomials in $\mathbb{C}(q,t)[z_1,\dots,z_{n+1}]$; here λ varies over the set of partitions of length at most $n+1$. These polynomials interpolate between several well–known families of symmetric polynomials such as the Schur polynomials $P_\lambda(z,0,0)$, the Hall-Littlewood polynomials $P_\lambda(z,0,t)$ and the Jack polynomials to name a few. The subject has deep connections with combinatorics, geometry and representation theory and there is a vast literature on the subject.

In this paper we shall be interested in the connection between the specialized Macdonald polynomials $P_\lambda(z,q,0)$ and the representation theory of the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. Such a connection was first shown to exist in [23] and we discuss this briefly. Fix a Borel subalgebra $\widehat{\mathfrak{b}}$ of $\widehat{\mathfrak{sl}}_{n+1}$, let Λ_i $0\le i\le n$ be a corresponding set of fundamental weights and let $V(\Lambda_i)$ be the associated integrable highest weight representation of $\widehat{\mathfrak{sl}}_{n+1}$. Given a partition λ or equivalently a dominant integral weight of $\mathfrak{sl}_{n+1}$ there exists an element w of the affine Weyl group and $0\le i\le n$ such that $w_\circ w\Lambda_i=(\lambda+\Lambda_0)$ where $w_\circ$ is the longest element of the Weyl group of $\mathfrak{sl}_{n+1}$. The main result of [23] shows that $P_\lambda(z,q,0)$ is the character of the $\widehat{\mathfrak{b}}$–module generated by the one-dimensional subspace $V_{w\Lambda_i}(\Lambda_i)$ of $V(\Lambda_i)$. This family of Demazure modules is special since the modules admit an action of the standard maximal parabolic subalgebra. (We remind the reader that the derived subalgebra of the standard maximal parabolic is also called the current algebra of $\mathfrak{sl}_{n+1}$; i.e., it is the subalgebra of $\widehat{\mathfrak{sl}}_{n+1}$ consisting of polynomial maps $\mathbb{C}\to\mathfrak{sl}_{n+1}$.) This result was later extended in [14] to twisted affine Lie algebras and the untwisted affine Lie algebras associated to a simply-laced simple Lie algebra. The result was known to be false for the affine algebras associated to the non-simply laced Lie algebras.

Recently it was shown in [5] that $P_\lambda(z,q,0)$ can in fact be realized as the character of a suitable module for the current algebra associated to a non-simply laced simple Lie algebra. This module has the corresponding Demazure module as a (possibly) proper quotient. One of the goals of this paper is to give explicit recurrences and closed formulae for the character of a level two Demazure module. One approach to the problem is to use the idea of Demazure flags which was first introduced and developed in [16]. It was further studied in [22] where it is shown that in the case of B_r, C_r, F_4 it suffices to study the relationship between level one and level two Demazure modules for affine $\mathfrak{sl}_{n+1}$. (In the case of G_2 one also has to understand level three modules for A_1; this was done in [1], [8], and we will say no more about it in this paper.) The level two modules are also modules for the current algebra, are indexed by dominant integral weights λ and are $\widehat{\mathfrak{b}}$-submodules of $V(\Lambda_i+\Lambda_j)$; here $0 \le i, j \le n$ and w are chosen so that $w_\circ w(\Lambda_i+\Lambda_j) = (\lambda+2\Lambda_0)$. As a consequence of the main result of this paper, we give explicit formula for the specialized Macdonald polynomials associated to root systems of type B, C, F in terms of the level one affine Demazure modules for the associated affine Lie algebra.

The level two Demazure modules for affine $\mathfrak{sl}_{n+1}$ also appear in a completely different context. They are the classical limit of an important family of modules for quantum affine $\mathfrak{sl}_{n+1}$. These modules occur in the work of [12], [13], and we refer the reader to Section 1.7 for further details. Not much is known about the structure of these modules for the quantum affine algebra. Our results show that their character is given by the polynomials $G_{0,\lambda}(z, q)$.

To understand the characters of level two Demazure modules it is best to work in a more general framework. Thus we define a family of (finite-dimensional) modules $M(\nu, \lambda)$ (see [24]) for the current algebra of $\mathfrak{sl}_{n+1}$ which are indexed by two dominant integral weights and interpolate between level one and level two Demazure modules. The graded characters of these modules are precisely the polynomials $G_{\nu,\lambda}(z, q)$. Although in this paper we only consider the case of $\mathfrak{sl}_{n+1}$ and level two, the definitions we give go through in a straightforward way to other simple Lie algebras and higher levels. However, the representation theory becomes much more difficult and is still relatively undeveloped.

The paper is organized as follows. In Section 1 we define the polynomials $G_{\nu,\lambda}(z, q)$ (see equations (1.3) and (1.4)) in terms of Macdonald polynomials (of type A_n) and state the main theorem. We explain in some detail in Section 1.7 the motivation and the methods used to prove the main results and discuss possible further directions. In Section 2 we introduce the modules $M(\nu, \lambda)$ and study their graded characters. In Section 2.4 we explain the key steps in deducing the main theorem of this paper. In the remaining two sections of the paper we prove the key step and give explicit formulae for the specialized Macdonald polynomials in term of the polynomials $G_{0,\lambda}$ and vice-versa.

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1. The main results

Let $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ be the set of complex numbers, integers, non–negative integers and positive integers respectively. For $i, j \in \mathbb{Z}$ with $i \le j$ we let $[i, j] = \{i, i+1, \dots, j\}$. Given an indeterminate q and $n, r \in \mathbb{Z}$ set

$$[n]_q = \frac{1-q^n}{1-q}, \quad \begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q[n-1]_q \dots [n-r+1]_q}{[1]_q[2]_q \dots [r]_q}, \quad 0 < r \le n,$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad n \in \mathbb{Z}_+, \quad \begin{bmatrix} n \\ r \end{bmatrix}_q = 0 \text{ unless } \{n, r, n-r\} \subset \mathbb{Z}_+.$$

1.1. Let $\mathfrak{g}$ be the simple Lie algebra $\mathfrak{sl}_{n+1}$ and let $\mathfrak{h}$ be a fixed Cartan subalgebra with basis elements $\{h_i : 1 \le i \le n\}$. Fix a set $\{\alpha_i : 1 \le i \le n\}$ of simple roots for $(\mathfrak{g}, \mathfrak{h})$ and a corresponding set $\{\omega_i : 1 \le i \le n\}$ of fundamental weights. It is convenient to set $\omega_0 = \omega_{n+1} = 0$. Let

$$R^+ = \{\alpha_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j : 1 \le i \le j \le n\},$$

be the set of positive roots. Let Q, P (resp. Q^+, P^+) be the $\mathbb{Z}$-span (resp. $\mathbb{Z}_+$-span) of the set of simple roots and fundamental weights respectively. Define a partial order $\le$ on P by: $\mu \le \lambda$ if and only if $\lambda - \mu \in Q^+$. Define functions $\mathrm{ht} : P \to \mathbb{Z}$ and $\mathrm{ht}_r : Q \to \mathbb{Z}$ by

$$\mathrm{ht} \sum_{i=1}^n r_i \omega_i = \sum_{i=1}^n r_i, \quad \mathrm{ht}_r \sum_{i=1}^n s_i \alpha_i = \sum_{i=1}^n s_i. \tag{1.1}$$

Let $(\ ,\ ) : P \times P \to \mathbb{Q}$ be the symmetric bilinear form defined by setting $(\omega_i, \alpha_j) = \delta_{i,j}$ and set

$$P^+(1) = \{\lambda \in P^+ : (\lambda, \alpha_i) \le 1 \text{ for all } 1 \le i \le n\}.$$

Equivalently,

$$P^+(1) = \{\omega_{i_1} + \cdots + \omega_{i_k}, \quad 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

In the rest of the paper we shall write (often without further mention) an element $\lambda \in P^+$ as a sum $\lambda = 2\lambda_0 + \lambda_1$ with $\lambda_0 \in P^+$ and $\lambda_1 \in P^+(1)$.

1.2. The polynomials $p_\lambda^\mu(q)$ and $G_{\nu,\lambda}(z, q)$

Given an indeterminate q and $\lambda, \mu \in P^+$ define $p_\lambda^\mu \in \mathbb{Z}_+[q]$ by

$$p_\lambda^\mu(q) = q^{\frac{1}{2}(\lambda+\mu_1,\ \lambda-\mu)} \prod_{j=1}^n \begin{bmatrix} (\lambda - \mu,\ \omega_j) + (\mu_0, \alpha_j) \\ (\lambda - \mu, \omega_j) \end{bmatrix}_q, \quad \mu = 2\mu_0 + \mu_1.$$

Notice that $p_\lambda^\lambda = 1$ and $p_\lambda^\mu = 0$ if $\lambda - \mu \notin Q^+$. Moreover,

$$(\lambda + \mu_1, \lambda - \mu) = (\lambda - \mu, \lambda - \mu) + 2(\mu - \mu_0, \lambda - \mu) \in 2\mathbb{Z}_+, \text{ if } \lambda - \mu \in Q^+,$$

and in particular $p_\lambda^\mu \in \mathbb{Z}_+[q]$ as asserted. It will be convenient to extend the definition by setting $p_\lambda^\mu = 0$ if λ or μ are in $P \setminus P^+$.

Given $\lambda \in P^+$ and a set of indeterminates $z = (z_1, \ldots, z_{n+1})$ let $s_\lambda(z)$ and $P_\lambda(z, q, 0)$ be the associated Schur polynomial and the specialized Macdonald polynomial respectively. Recall that the Schur polynomials are a basis for the ring of symmetric polynomials $\mathbb{C}[z_1, \ldots, z_{n+1}]$ while the specialized Macdonald polynomials are a basis for the ring of symmetric polynomials in $\mathbb{C}(q)[z_1, \ldots, z_{n+1}]$ (see [21]).

Define elements $G_\lambda(z, q) \in \mathbb{C}[q][z_1, \ldots, z_{n+1}]$ recursively by requiring,

$$P_\lambda(z, q, 0) = \sum_{\mu \in P^+} p_\lambda^\mu(q) G_\mu(z, q). \tag{1.2}$$

Since $p_\lambda^\mu = 0$ if $\mu \not\leq \lambda$, it follows that

$$P_{\omega_i}(z, q, 0) = G_{\omega_i}(z, q) = s_{\omega_i}(z), \quad i \in [0, n], \quad P_\lambda(z, q, 0) = G_\lambda(z, q) + \sum_{\mu < \lambda} p_\lambda^\mu G_\mu(z, q).$$

A straightforward induction on the partial order on P^+ proves that $G_\lambda(z, q) \in \mathbb{C}[q][z]$ and so the elements $\{G_\lambda(z, q) : \lambda \in P^+\}$ are also a basis for the ring of symmetric polynomials in $\mathbb{C}(q)[z_1, \ldots, z_{n+1}]$. Hence there exists a subset $\{a_\lambda^\mu : \mu \in P^+\} \subset \mathbb{C}(q)$ such that

$$a_\lambda^\lambda = 1, \quad a_\lambda^\mu = 0 \ \text{ if } \ \lambda - \mu \notin Q^+,$$

satisfying

$$G_\lambda(z, q) = \sum_{\mu \in P^+} a_\lambda^\mu(q) P_\mu(z, q, 0) \quad \text{and} \qquad \sum_{\mu \in P^+} a_\lambda^\mu p_\mu^\nu = \delta_{\lambda,\mu} = \sum_{\mu \in P^+} p_\lambda^\mu a_\mu^\nu. \tag{1.3}$$

Given a pair of elements $\nu, \lambda \in P^+$ set

$$G_{\nu,\lambda}(z, q) = \sum_{\mu \in P^+} q^{(\lambda+\nu-\mu,\nu)} a_\lambda^{\mu-\nu}(q) P_\mu(z, q, 0), \tag{1.4}$$

where we understand that $a_\lambda^{\mu-\nu} = 0$ if $\mu - \nu \notin P^+$. Notice that

$$G_{\nu,0}(z, q) = P_\nu(z, q, 0), \quad G_{0,\lambda}(z, q) = G_\lambda(z, q).$$

Here the second equality is obvious while the first equality follows since $a_0^0 = 1$ and $a_0^{\mu-\nu} \neq 0$ only if $\mu - \nu \in P^+$ and $\nu - \mu \in Q^+$ which is possible only if $\nu = \mu$.

1.3. The main result

For $\mu \in P^+ \setminus \{0\}$, set

$$\max \mu = \max\{i \in [1, n] : \mu(h_i) > 0\}, \quad \min \mu = \min\{i \in [i, n] : \mu(h_i) > 0\}.$$

In what follows we shall understand that if $\mu = 0$ then $\max \mu = \min \mu = 0$. We say that a pair $(\nu, \lambda) \in P^+ \times P^+$ is *compatible* if one of the following hold: writing $\nu = 2\nu_0 + \nu_1$, either,
• $\lambda_1 = 0$, or
• $\lambda_1 \neq 0$ in which case we must have $\nu_0 = \omega_i$ for some $i \in [0, n]$ and $\max \nu_1 < \min \lambda_1$. If in addition $i \geq 1$ (equivalently $\nu_0 \neq 0$) then we require $i < \min \lambda_1 - 1$ and $\nu_1(h_i) = \nu_1(h_{i+1}) = 0$.

A symmetric polynomial in $\mathbb{C}(q)[z_1, \dots, z_{n+1}]$ is Schur-positive if it is in the $\mathbb{Z}_+[q]$ span of $\{s_\mu : \mu \in P^+\}$. The following is one of the main results of this paper.

Theorem 1. *For a compatible pair $(\nu, \lambda) \in P^+ \times P^+$ the polynomial $G_{\nu,\lambda}(z, q)$ is Schur positive.*

We also give a closed formula for the polynomials a^μ_λ. Unlike the polynomial p^μ_λ which is non–zero for all $\mu \in P^+$ with $\mu \leq \lambda$, the polynomials a^μ_λ are non–zero on a much smaller set and the statement requires some additional definitions; the formulae can be found in Section 3 (see also Section 2.4 and equations (3.6), (3.8)).

1.4. To prove Theorem 1, we introduce a family of graded finite–dimensional modules $M(\nu, \lambda)$ for the standard maximal parabolic subalgebra of affine $\mathfrak{sl}_{n+1}$. These interpolate between the local Weyl module $M(\nu, 0) = W_{\rm loc}(\nu)$ and the level two Demazure module $M(0, \lambda) = D(2, \lambda)$. The graded characters of the modules $M(\nu, 0)$, $\nu \in P^+$ (resp. $M(0, \lambda)$, $\lambda \in P^+$) are linearly independent and their $\mathbb{C}[q]$-span contains the graded character of $M(\nu, \lambda)$. The representation theoretic version of Theorem 1 is the following. Given two graded $\mathfrak{g}[t]$-modules M_s, $s = 1, 2$, let $M_1 * M_2$ denote the fusion product (or graded tensor product) defined in [11]; we have included the definition in Section 2.7 for the reader's convenience.

Theorem 2. *Suppose that $(\nu, \lambda) \in P^+ \times P^+$ is a compatible pair. Then,*

$$\mathrm{ch}_{\rm gr}\, M(\nu, \lambda) = G_{\nu,\lambda}(z, q). \tag{1.5}$$

Moreover, we have an isomorphism of $\mathfrak{g}[t]$-modules

$$M(\nu, \lambda) \cong M(\nu, 0) * M(0, \lambda).$$

Corollary. *For (ν, λ) is compatible, we have*

$$G_{\nu,\lambda}(z, 1) = G_{\nu,0}(z, 1) G_{0,\lambda}(z, 1). \tag{1.6}$$

Proof of Corollary. Regarded as a $\mathfrak{g}$–module it follows from the preceding theorem and the definition of the fusion product that

$$M(\nu,\lambda) \cong_{\mathfrak{g}} M(\nu,0) \otimes M(0,\lambda).$$

On the other hand the theorem also gives that $G_\nu(z,1)$ is precisely the $\mathfrak{g}$-character of $M(\nu,\lambda)$. The corollary is now immediate. □

1.5. Outline of Proof of Theorem 2

Using the linear independence of the characters, we can write

$$\sum_{\mu\in P^+} h^\mu_{\nu,\lambda}\, \mathrm{ch}_{\mathrm{gr}}\, M(0,\mu) = \mathrm{ch}_{\mathrm{gr}}\, M(\nu,\lambda) = \sum_{\mu\in P^+} g^\mu_{\nu,\lambda}(q)\, \mathrm{ch}_{\mathrm{gr}}\, M(\mu,0),$$
$$g^\mu_{\nu,\lambda}(q), h^\mu_{\nu,\lambda}(q) \in \mathbb{C}[q].$$

In Proposition 2.5 we establish certain short exact sequences involving the modules $M(\nu,\lambda)$ with (ν,λ) compatible. The second assertion of Theorem 2 is established in Corollary 2.8.

To prove (1.5) we use the results of Section 2 and show in Lemma 3.1 that the polynomials $g^\mu_{\nu,\lambda}$ and $h^\mu_{\nu,\lambda}$ for (ν,λ) compatible satisfy certain recursions. In Section 3 and Section 4 we solve these recursions and in particular obtain that

$$h^u_{\nu,0} = p^\mu_\nu, \quad g^\mu_{\nu,\lambda} = q^{(\nu,\lambda+\nu-\mu)} g^{\mu-\nu}_{0,\lambda}.$$

Together with the results in [6], [23] this leads (see Section 2.4) to a proof of the equality in (1.5) and then the proof of Theorem 1 is essentially immediate.

1.6. Examples

We give some explicit examples of $G_{\nu,\lambda}(z,q)$ in terms of Schur polynomials.
Suppose that $\mathfrak{g} = \mathfrak{sl}_3$. Then,

$$\begin{aligned}
G_{0,\omega_1+\omega_2} &= s_{\omega_1+\omega_2},\\
G_{0,2\omega_1+2\omega_2} &= s_{2\omega_1+2\omega_2} + qs_{\omega_1+\omega_2} + q^2 s_0,\\
G_{0,2\omega_1+3\omega_2} &= s_{2\omega_1+3\omega_2} + q[2]_q s_{\omega_1+2\omega_2} + q^2 s_{2\omega_1} + q^2[2]_q s_{\omega_2} + qs_{3\omega_1+\omega_2},\\
G_{0,3\omega_1+3\omega_2} &= s_{3\omega_1+3\omega_2} + q(s_{4\omega_1+\omega_2} + s_{\omega_1+4\omega_2}) + (2q^2+q)s_{2\omega_1+2\omega_2}\\
&\quad + q^2[2]_q(s_{3\omega_1} + s_{3\omega_2}) + (q[2]_q)^2 s_{\omega_1+\omega_2} + q^3 s_0,\\
G_{\omega_1+\omega_2,2\omega_2} &= s_{\omega_1+3\omega_2} + qs_{2\omega_1+\omega_2} + q^2 s_{\omega_1} + q[2]_q s_{2\omega_2}.
\end{aligned}$$

Suppose that $\mathfrak{g} = \mathfrak{sl}_4$. Then,

$$\begin{aligned}
G_{0,\omega_1+\omega_2+\omega_3} &= s_{\omega_1+\omega_2+\omega_3} + qs_{\omega_2},\\
G_{0,2\omega_1+\omega_2+\omega_3} &= s_{2\omega_1+\omega_2+\omega_3} + q(s_{\omega_1+2\omega_3} + s_{\omega_1+\omega_2}) + q^2 s_{\omega_3}.
\end{aligned}$$

Suppose that $\mathfrak{g} = \mathfrak{sl}_5$. Then,

$$\begin{aligned} G_{0,\omega_1+\omega_4} &= s_{\omega_1+\omega_4}, \\ G_{0,2\omega_1+2\omega_4} &= s_{2\omega_1+2\omega_4} + qs_{\omega_1+\omega_4} + q^2 s_0, \\ G_{0,\omega_1+\omega_2+\omega_4} &= s_{\omega_1+\omega_2+\omega_4} + qs_{\omega_2}. \end{aligned}$$

We give some examples where $G_{\nu,\lambda}$ is Schur positive for non-compatible pairs (ν, λ). In the case of $\mathfrak{sl}_3$ we have

$$G_{2\omega_1,\omega_1+\omega_2} = s_{3\omega_1+\omega_2} + q[2]_q(s_{\omega_1+2\omega_2} + s_{2\omega_1} + qs_{\omega_2}).$$

In the case of $\mathfrak{sl}_4$ we have

$$\begin{aligned} G_{\omega_3,\omega_1+\omega_2+\omega_3} &= s_{\omega_1+\omega_2+2\omega_3} + q(s_{2\omega_1+\omega_3} + s_{\omega_1+2\omega_2}) + q^2 s_{\omega_1} + q[2]_q s_{\omega_2+\omega_3}, \\ G_{\omega_2,\omega_1+\omega_2+\omega_3} &= s_{\omega_1+2\omega_2+\omega_3} + q(s_{2\omega_1+2\omega_3} + s_{2\omega_1+\omega_2} + s_{\omega_2+2\omega_3}) \\ &\quad + q[2]_q s_{2\omega_2} + 2q^2 s_{\omega_1+\omega_3} + q^3 s_0, \\ G_{\omega_1,\omega_1+\omega_2+\omega_3} &= s_{2\omega_1+\omega_2+\omega_3} + q(s_{2\omega_2+\omega_3} + s_{\omega_1+2\omega_3}) + q[2]_q s_{\omega_1+\omega_2} + q^2 s_{\omega_3}. \end{aligned}$$

For $\mathfrak{sl}_5$, we have

$$\begin{aligned} G_{\omega_1,\omega_1+\omega_4} &= s_{2\omega_1+\omega_4} + q(s_{\omega_2+\omega_4} + s_{\omega_1}), \\ G_{\omega_1+\omega_2,\omega_1+\omega_4} &= s_{2\omega_1+\omega_2+\omega_4} + q(s_{3\omega_1} + s_{2\omega_1+\omega_4}) + q[2]_q(s_{\omega_1+\omega_3+\omega_4} + qs_{\omega_3}) \\ &\quad + q^3 s_{2\omega_4} + (2q^2 + q)s_{\omega_1+\omega_2}. \end{aligned}$$

We also give some explicit examples of $G_{\nu,\lambda}(z, q)$ (and hence also formulae for a^μ_λ) in terms of Macdonald polynomials in the case of $\mathfrak{g} = \mathfrak{sl}_3$. Thus, we have

$$G_{0,\omega_1+\omega_2}(z,q) = P_{\omega_1+\omega_2}(z,q,0) - qP_0(z,q), \text{ i.e., } a^{\omega_1+\omega_2}_{\omega_1+\omega_2} = 1, \quad a^0_{\omega_1+\omega_2} = -q,$$
$$G_{\omega_1+\omega_2,2\omega_2}(z,q) = P_{\omega_1+3\omega_2}(z,q) - q^2 P_{2\omega_1+\omega_2}(z,q), \quad \text{i.e. } a^{\omega_1+3\omega_2}_{2\omega_2} = 1, \quad a^{2\omega_1+\omega_2}_{2\omega_2} = -q^2.$$

Note that the a^μ_λ alternates in sign.

1.7. We explain in some detail the motivation for these results and assume for just this discussion, that $\mathfrak{g}$ is an arbitrary simple Lie algebra. As discussed in the introduction, it was shown in [14,23] that the specialized Macdonald polynomial $P_\lambda(z, q, 0)$ is the character of a Demazure module occurring in a fundamental integrable highest weight representation of the affine Lie algebra associated to a simple Lie algebra of type A, D, E. The graded characters of Demazure modules occurring in higher level integrable representations are not very well understood. There are combinatorial results for weights of the form $r\lambda$ for some $r \in \mathbb{Z}_+$ and λ a dominant integral weight for $\mathfrak{g}$ (see for instance

[17–20]). However, in this paper we are interested in the study of level two Demazure modules which are *not* of this type. This interest arises from the connections with cluster algebras and monoidal categorification (see [2,12,13]). It was shown in [3] that the classical limit $q \to 1$ of the representations corresponding to a cluster variable in a cluster algebra of type A_n is a level two Demazure module. This is one motivation for our study since our results show that the character of the irreducible representation of the quantum affine algebra is given by the polynomials $G_{0,\lambda}(q,z)$, $\lambda \in P^+(1)$.

Another reason to be interested in this problem comes from the connection with Macdonald polynomials associated to root systems of simple Lie algebras of type B, C, F, G. For these algebras it has long been known that the Macdonald polynomial is "too big" to be the character of the Demazure module. It was shown recently in [5] that for these algebras the Macdonald polynomial is the character of a family of modules called the local Weyl modules. These modules denoted $W_{\text{loc}}(\lambda)$, are defined for all simple Lie algebras and are indexed by the dominant integral weights of the underlying simple Lie algebra. Like the Demazure modules they are modules for the standard maximal parabolic subalgebra of the affine Lie algebra or the current algebra. In types A, D, E the Weyl and Demazure modules coincide (see [6], [11]), but in other types the corresponding Demazure module can be a proper quotient of the local Weyl module.

A study of the relationship between the local Weyl modules and the level one Demazure modules in the non-simply laced case was initiated by K. Naoi in [22] and we now discuss his results. Let μ be a dominant integral weight for a simple Lie algebra of type X_n and $D(1,\mu)$ a Demazure module occurring in a level one integrable highest weight representation of the affine Lie algebra of type $X_n^{(1)}$ where $X \in \{B, C, F\}$. Naoi proved that if μ is zero on the short simple roots then the local Weyl module is isomorphic to a level one Demazure module. Otherwise, he showed that the local Weyl module admits a non-trivial flag whose successive quotients are isomorphic to level one Demazure modules. The modules in question are all graded and hence one can define the graded multiplicity of the module $D(1,\mu)$ in $W_{\text{loc}}(\lambda)$, denoted $[W_{\text{loc}}(\lambda) : D(1,\mu)]$. This multiplicity is obviously zero if $\mu \nleq \lambda$.

Naoi's next result related this graded multiplicity to a problem in type A. Let $\mu_s \in P$ be the element defined by

$$\mu_s(h_i) = \begin{cases} 0 & \text{if } \alpha_i \text{ is a long root,} \\ \mu(h_i) & \text{if } \alpha \text{ is a short root.} \end{cases}$$

We also regard μ_s as a dominant integral weight for the simple Lie algebra $\mathfrak{g}_s$ which is generated by the short simple roots. Note that $\mathfrak{g}_s$ is of type A_1 if $X_n = B_n$, of type A_2 if $X_4 = F_4$ and of type A_{n-1} if $X_n = C_n$. Let $W^s_{\text{loc}}(\mu_s)$ and $D^s(2,\mu_s)$ be the local Weyl module (equivalently the level one Demazure module) and the level two Demazure module associated to the untwisted affine Lie algebra $\widehat{\mathfrak{g}}_s$. Naoi proved that the module $W^s_{\text{loc}}(\lambda_s)$ admits a flag whose successive quotients are level two Demazure modules $D^s(2,\mu_s)$ and moreover,

$$[W_{\rm loc}(\lambda):D(1,\mu)]=\delta_{\lambda-\lambda_s,\mu-\mu_s}[W^s_{\rm loc}(\lambda_s):D^s(2,\mu_s)].$$

Together with Naoi's results, Theorem 1 of this paper can be reformulated as asserting the following: if $\mathfrak{g}$ is of type B, C, F then

$$P_\lambda(z,q,0)=\sum_{\mu\le\lambda,\mu\in P^+}\delta_{\mu-\mu_s,\lambda-\lambda_s}p^{\mu_s}_{\lambda_s}\,{\rm ch}_{\rm gr}\,D(1,\mu).$$

The preceding formula is clearly invertible since the transition matrix (between the basis of the local Weyl modules and the Demazure modules) is upper triangular with all diagonal entries being 1. In particular our result gives a formula for the graded character of the level one Demazure module when $\mathfrak{g}$ is not simply laced in terms of Macdonald polynomials. This finally completes the picture begun in [14,15,23].

1.8. There are some obvious questions that can be raised. For instance, one could define the polynomials $G_{\nu,\lambda}$ in the same way for other simple Lie algebras and ask if they arise as the graded character of some module. Another question would be to ask if there is some general framework in which to study polynomials which interpolate between characters of higher level Demazure modules. In fact, one can define suitable analogs of the modules $M(\nu,\mu)$ to address these questions. In this paper, the input coming from the representation theory of quantum affine $\mathfrak{sl}_{n+1}$ [2,3,12,13] has been critical. The analogs of these results are not known, however, and are not easily formulated for the other quantum affine algebras or for higher level Demazure modules (see however [4]). We hope to return to these ideas elsewhere.

2. The modules $M(\nu,\lambda)$

We recall the definition of the current algebra associated to $\mathfrak{sl}_{n+1}$. We also recall several results on their representation theory which are needed for our study. We then introduce the modules $M(\nu,\lambda)$ and explain in Section 2.4 how the study of these modules leads to the proof of Theorem 1. In the rest of the section we study the graded characters of these modules and show that they satisfy certain recursions.

2.1. The current algebra $\mathfrak{sl}_{n+1}[t]$

Let t be an indeterminate and $\mathbb{C}[t]$ the polynomial ring with complex coefficients. Denote by $\mathfrak{g}[t]$ the Lie algebra with underlying vector space $\mathfrak{g}\otimes\mathbb{C}[t]$ and commutator given by

$$[a\otimes f,b\otimes g]=[a,b]\otimes fg,\quad a,b\in\mathfrak{g},\quad f,g\in\mathbb{C}[t].$$

Then $\mathfrak{g}[t]$ and its universal enveloping algebra admit a natural $\mathbb{Z}_+$-grading given by declaring a monomial $(a_1\otimes t^{r_1})\dots(a_p\otimes t^{r_p})$ to have grade $r_1+\ldots+r_p$, where $a_s\in\mathfrak{g}$ and $r_s\in\mathbb{Z}_+$ for $1\le s\le p$. We freely identify the subspace $\mathfrak{g}\otimes 1$ with the Lie algebra $\mathfrak{g}$.

We shall be interested in the category of finite–dimensional $\mathbb{Z}$–graded modules for $\mathfrak{g}[t]$. An object of this category is a finite-dimensional module V for $\mathfrak{g}[t]$ with a compatible $\mathbb{Z}$–grading,

$$V = \bigoplus_{s\in\mathbb{Z}} V[s], \quad (x\otimes t^r)V[s] \subset V[r+s], \quad x\in\mathfrak{g}, \quad r\in\mathbb{Z}_+.$$

For $s\in\mathbb{Z}$, clearly $V[s]$ is a $\mathfrak{g}$–module. For any $p\in\mathbb{Z}$ let τ_p^*V be the graded $\mathfrak{g}[t]$–module which is given by shifting all the grades up by p and leaving the action of $\mathfrak{g}[t]$ unchanged. The morphisms between graded modules are $\mathfrak{g}[t]$–module maps of grade zero.

Let $\mathbb{Z}[q,q^{-1}][P]$ be the group ring of P with basis $\{e(\mu):\mu\in P\}$ and coefficients in $\mathbb{Z}[q,q^{-1}]$. Any finite-dimensional $\mathfrak{g}$-module V can be written as,

$$V = \bigoplus_{\mu\in P} V_\mu, \quad V_\mu = \{v\in V : hv = \mu(h)v, \quad h\in\mathfrak{h}\},$$

and the character of a $\mathfrak{g}$–module V is the element $\operatorname{ch} V = \sum_{\mu\in P}\dim V_\mu\, e(\mu)$ of $\mathbb{Z}[P]$. The irreducible finite–dimensional $\mathfrak{g}$-modules are indexed by elements of P^+, and given $\lambda\in P^+$ we let $V(\lambda)$ be an irreducible module corresponding to λ. Setting $z_i = e(\omega_i-\omega_{i-1})$, $1\le i\le n+1$, a classical result asserts that $\operatorname{ch} V(\lambda)$ is just the Schur polynomial $s_\lambda(z)$. If V is a graded $\mathfrak{g}[t]$–module, let

$$\operatorname{ch}_{\mathrm{gr}} V = \sum_{s\in\mathbb{Z}} q^s \operatorname{ch} V[s] = \sum_{\mu\in P^+}\left(\sum_{s\in\mathbb{Z}} \dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu),V[s])q^s\right)\operatorname{ch} V(\mu).$$

2.2. The modules $M(\nu,\lambda)$

We recall the definition of a family of modules $M(\nu,\lambda)$ which were introduced and studied in [24]. Fix a Chevalley basis $\{x^\pm_{i,j}, h_i : 1\le i\le j\le n\}$ for $\mathfrak{g}$ and let $x^\pm_{j,j} = x^\pm_j$, $h_{i,j} = h_i+\ldots+h_j$ for all $1\le i\le j\le n$.

For $\nu,\lambda\in P^+$ with $\lambda = 2\lambda_0+\lambda_1$, let $M(\nu,\lambda)$ be the $\mathfrak{g}[t]$–module generated by an element $w_{\nu,\lambda}$ with the following relations:

$$(x_i^+\otimes 1)w_{\nu,\lambda} = 0, \ \ (h\otimes t^r)w_{\nu,\lambda} = \delta_{r,0}(\lambda+\nu)(h)w_{\nu,\lambda}, \ \ (x_i^-\otimes 1)^{(\lambda+\nu)(h_i)+1}w_{\nu,\lambda} = 0, \tag{2.1}$$

$$(x_\alpha^-\otimes t^{\nu(h_\alpha)+\lceil\lambda(h_\alpha)/2\rceil})w_{\nu,\lambda} = 0, \tag{2.2}$$

for all $i\in[1,n]$, $h\in\mathfrak{h}$, $r\in\mathbb{Z}_+$ and $\alpha\in R^+$. An inspection of the defining relations shows that

$$M(\omega_i,0)\cong_{\mathfrak{g}[t]} M(0,\omega_i)\cong_{\mathfrak{g}} V(\omega_i), \quad M(0,2\omega_i)\cong_{\mathfrak{g}} V(2\omega_i), \quad i\in[1,n] \tag{2.3}$$

$$M(\nu+\omega_i,2\lambda_0)\cong_{\mathfrak{g}[t]} M(\nu,2\lambda_0+\omega_i). \tag{2.4}$$

Since the defining relations of $M(\nu,\lambda)$ are graded, it follows that the module $M(\nu,\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]$–module once we declare the grade of $w_{\nu,\lambda}$ to be zero. In the case when $\lambda = 0$ it is known (see [7]) that the relations in (2.2) are a consequence of the relations in (2.1). In particular the module $M(\nu,0)$ is just the local Weyl module, which is usually denoted as $W_{\rm loc}(\nu)$ and is defined by the relations given in (2.1) with $\lambda = 0$. The local Weyl modules are known (see [7]) to be finite–dimensional and since $M(\nu,\lambda)$ is obviously a quotient of $W_{\rm loc}(\nu+\lambda)$, it follows that $M(\nu,\lambda)$ is also finite–dimensional. Moreover standard arguments show that

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), M(\nu,\lambda)) \neq 0 \implies \nu+\lambda-\mu \in Q^+,$$
$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\nu+\lambda), M(\nu,\lambda)) = 1.$$

This discussion shows that the set $\{\operatorname{ch}_{\rm gr} M(\mu,0) : \mu \in P^+\}$ (resp. the set $\{\operatorname{ch}_{\rm gr} M(0,\mu) : \mu \in P^+\}$) is linearly independent and that the $\mathbb{Z}[q,q^{-1}]$–span of the set contains $\operatorname{ch} V(\lambda)$, $\lambda \in P^+$. It follows that we can write,

$$\operatorname{ch}_{\rm gr} M(\nu,\lambda) = \sum_{\mu\in P^+} g^{\mu}_{\nu,\lambda}(q) \operatorname{ch}_{\rm gr} M(\mu,0) = \sum_{\mu\in P^+} h^{\mu}_{\nu,\lambda}(q) \operatorname{ch}_{\rm gr} M(0,\mu), \tag{2.5}$$

for some $g^{\mu}_{\nu,\lambda}, h^{\mu}_{\nu,\lambda} \in \mathbb{C}[q]$ satisfying,

$$g^{\nu+\lambda}_{\nu,\lambda} = 1 = h^{\nu+\lambda}_{\nu,\lambda}, \quad g^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu,\lambda} = 0 \ \text{ if } \ \lambda+\nu-\mu \notin Q^+. \tag{2.6}$$

(In what follows we shall assume that $g^{\nu}_{\lambda,\mu} = 0$ if one of λ,μ,ν are not in P^+.) The linear independence of the characters implies that for all $\nu,\mu \in P^+$,

$$\sum_{\mu'\in P^+} h^{\mu'}_{\nu,0} g^{\mu}_{0,\mu'} = \delta_{\nu,\mu} = \sum_{\mu'\in P^+} g^{\mu'}_{0,\nu} h^{\mu}_{\mu',0}. \tag{2.7}$$

It was shown in [6] that $W_{\rm loc}(\nu)$ (or equivalently $M(\nu,0)$) is graded isomorphic to a Demazure module occurring in a level one representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. Using [23] it follows that

$$\operatorname{ch}_{\rm gr} M(\lambda,0) = P_\lambda(z,q,0),$$

and hence we have

$$\operatorname{ch}_{\rm gr} M(\nu,\lambda) = \sum_{\mu\in P^+} g^{\mu}_{\nu,\lambda} P_\mu(z,q,0). \tag{2.8}$$

Remark. The more usual definition of the Demazure modules in the affine case is by using an element of the affine dominant integral weights and an element of the affine Weyl group. To make the connection with this definition, let $w_\circ$ be the longest element of the finite Weyl group and let Λ_0 be the affine dominant integral weight associated to the zero node of the affine Dynkin diagram. Choose w in the affine Weyl group such that the weight $\nu+\Lambda_0 = w_\circ w\Lambda_i$ where Λ_i is a fundamental weight. Then $M(\nu, 0)$ corresponds to the Demazure module $V_w(\Lambda_i)$.

It is known (see [9] and the references in that paper) that the module $M(0,\lambda)$ is isomorphic to a Demazure module (denoted in the literature as $D(2,\lambda)$) occurring in a level two representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1}$. In this case one makes the connection with the usual definition of Demazure modules by choosing an element w of the affine Weyl group which makes $\lambda + 2\Lambda_0 = w_\circ w(\Lambda_i + \Lambda_j)$ here $1 \leq i, j \leq n$.

2.3. The following is the key result needed to prove the main theorem.

Proposition. *For $\lambda, \nu, \mu \in P^+$ with (ν, λ) compatible, we have*

$$g^{\mu}_{\nu,\lambda} = q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{0,\lambda} \qquad h^{\mu}_{\nu,0} = p^{\mu}_{\nu}. \tag{2.9}$$

The proof of the first equality can be found in Section 3.2 while the proof of the second equality is in Section 4.

2.4. Proof of equality (1.5) *and Theorem 1*

Using the second equality in Proposition 2.3 and equations (1.3), (2.5) we get

$$\sum_{\mu\in P^+} p^{\mu}_{\nu} G_{\mu}(z,q) = P_{\nu}(z,q,0) = \operatorname{ch}_{\mathrm{gr}} M(\nu,0) = \sum_{\mu\in P^+} p^{\mu}_{\nu} \operatorname{ch}_{\mathrm{gr}} M(0,\mu).$$

Since $M(\omega_i, 0) \cong V(\omega_i)$ we have $G_{\omega_i}(z,q) = \operatorname{ch}_{\mathrm{gr}} M(0,\omega_i)$ and an induction on the partial order on P^+ (recall that the coefficient matrix is invertible) proves that

$$G_{\mu}(z,q) = \operatorname{ch}_{\mathrm{gr}} M(0,\mu) \ \text{ for all } \mu \in P^+.$$

Using (1.3) and (2.5) again we have

$$\sum_{\mu\in P^+} a^{\mu}_{\lambda} P_{\mu}(z,q,0) = G_{\lambda}(z,q) = \sum_{\mu\in P^+} g^{\mu}_{0,\lambda} \operatorname{ch}_{\mathrm{gr}} M(\mu,0) = \sum_{\mu\in P^+} g^{\mu}_{0,\lambda} P_{\mu}(z,q,0).$$

The linear independence of the Macdonald polynomials implies that

$$a^{\mu}_{\lambda} = g^{\mu}_{0,\lambda}, \quad \text{for all} \quad \mu \in P^+. \tag{2.10}$$

Using the first equality in Proposition 2.3 and equation (1.4) we now have

$$G_{\nu,\lambda}(z,q) = \operatorname{ch}_{\mathrm{gr}} M(\nu,\lambda) \ \text{ if } \ (\nu,\lambda) \in P^+ \times P^+ \ \text{ compatible.}$$

Since the modules $M(\nu, \lambda)$ are finite–dimensional and each graded piece of the module is a $\mathfrak{g}$–module it follows that $G_{\nu,\lambda}(z,q)$ is Schur positive.

2.5. The proof of Proposition 2.3 requires a deeper understanding of the modules $M(\nu,\lambda)$ and we now state the key representation theoretic result of this paper.

Proposition.

(i) Let $(\nu,\lambda)\in P^+\times P^+$ be a compatible pair.

(a) If $\nu(h_j)\geq 2$ for some $j\in[1,n]$ there exists an exact sequence of $\mathfrak{g}[t]$–modules,

$$0\to\tau^*_{(\lambda_0+\nu,\alpha_j)-1}M(\nu-\alpha_j,\lambda)\to M(\nu,\lambda)\to M(\nu-2\omega_j,\lambda+2\omega_j)\to 0.$$

(b) If $\nu\in P^+(1)$ with $\max\nu<\min\lambda_1=m$ and $p=\min(\lambda_1-\omega_m)>0$ we have an exact sequence of $\mathfrak{g}[t]$–modules,

$$0\to\tau^*_{\frac{1}{2}(\lambda,\alpha_{m,p})}M(\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1})\to M(\nu+\omega_m,\lambda-\omega_m)$$
$$\to M(\nu,\lambda)\to 0.$$

(ii) If $\lambda\in P^+(1)$ and $m\in[1,n]$ with $m<\min\lambda$, we have an exact sequence of $\mathfrak{g}[t]$-modules,

$$0\to\tau_1^*M(\omega_{m-1},\lambda+\omega_{m+1})\to M(\omega_m,\lambda+\omega_m)\to M(0,\lambda+2\omega_m)\to 0.$$

2.6. From now on we prove Proposition 2.5. As a first step we establish the existence of the appropriate right exact sequences. Part (i) was proved in [24] and we just indicate the main steps. For part (a) notice that since (ν,λ) is compatible it follows that if $\nu(h_j)\geq 2$ then either $\lambda_1=0$ or $\min\lambda_1>j+1$. In this case the existence of the map $M(\nu,\lambda)\to M(\nu-2\omega_j,\lambda+2\omega_j)\to 0$ is trivial from the defining relations and also the fact that $(x_j^-\otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu,\lambda}$ is in the kernel of the map. One then proves that this element generates the kernel and that there exists a well-defined map

$$M(\nu-\alpha_j,\lambda)\twoheadrightarrow\mathbf{U}(\mathfrak{g}[t])(x_j^-\otimes t^{(\nu+\lambda_0,\alpha_j)-1})w_{\nu,\lambda}.$$

The proof of part (b) is similar. This time one proves that there is a canonical map $M(\nu+\omega_m,\lambda-\omega_m)\to M(\nu,\lambda)\to 0$ whose kernel is generated by $(x^-_{m,p}\otimes t^{\frac{1}{2}(\lambda,\alpha_{m,p})})w_{\nu+\omega_m,\lambda-\omega_m}$. One then proves that there exists a well-defined map

$$M(\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1})\twoheadrightarrow\mathbf{U}(\mathfrak{g}[t])(x^-_{m,p}\otimes t^{\frac{1}{2}(\lambda,\alpha_{m,p})})w_{\nu+\omega_m,\lambda-\omega_m}.$$

The proof of part (ii) is similar, but since this is not in the literature, we provide all the details for the reader's convenience. An inspection of the defining relations shows the existence of a surjective map of $\mathfrak{g}[t]$-modules

$$M(\omega_m,\lambda+\omega_m)\to M(0,\lambda+2\omega_m)\to 0,$$

whose kernel is generated by the elements

$$\left(x_\alpha^- \otimes t^{\lceil(\lambda+\omega_m)(h_\alpha)/2\rceil}\right) w_{\omega_m,\lambda+\omega_m}, \quad \alpha \in R^+, \quad \lambda(h_\alpha) \in 2\mathbb{Z}_+, \quad \omega_m(h_\alpha) = 1. \tag{2.11}$$

Taking $\alpha = \alpha_m$ we see that $(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m}$ is in the kernel and we claim that it generates the kernel. For the claim, let $\alpha \in R^+$ be as in (2.11) and write

$$\alpha = \beta + \alpha_m + \gamma, \quad \beta \in \sum_{i=1}^{m-1} \mathbb{Z}_+\alpha_i, \quad \gamma \in \sum_{i=m+1}^{n} \mathbb{Z}_+\alpha_i.$$

If $\beta, \gamma \in R^+$ then $(\lambda + \omega_m)(h_\beta) = 0$ and $(\lambda + \omega_m)(h_\gamma) = \lambda(h_\gamma)$; hence the defining relations of $M(\nu, \lambda)$ give

$$(x_\beta^- \otimes 1)(x_\gamma^- \otimes t^{\lceil\lambda(h_\gamma)/2\rceil})(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = (x_\alpha^- \otimes t^{\lceil(\lambda+\omega_m)(h_\alpha)/2\rceil})w_{\omega_m,\lambda+\omega_m}.$$

Otherwise either $\beta = 0$ or $\gamma = 0$ and an identical argument gives the claim in this case.

We now show that the existence of the morphism

$$\tau_1^* M(\omega_{m-1}, \lambda + \omega_{m+1}) \to M(\omega_m, \lambda + \omega_m), \quad w_{\omega_{m-1},\lambda+\omega_{m+1}} \mapsto (x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m}.$$

The proof that $(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m}$ satisfies the relations in (2.1) is elementary. It remains to show that

$$(x_\alpha^- \otimes t^{\omega_{m-1}(h_\alpha)+\lceil(\lambda+\omega_{m+1})(h_\alpha)/2\rceil})(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = 0 \tag{2.12}$$

and for this we consider several cases. If $\alpha = \alpha_m$ the relations

$$(h_m \otimes 1)(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = (\lambda + 2\omega_m - \alpha_m)(h_m)(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = 0$$

and

$$(x_m^+ \otimes 1)(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = 0$$

imply that

$$(x_m^- \otimes 1)(x_m^- \otimes t)w_{\omega_m,\lambda+\omega_m} = 0$$

since $M(\omega_m, \lambda + \omega_m)$ is finite-dimensional $\mathfrak{g}$-module.

If α is in the span of $\{\alpha_j : 1 \le j < m-1\}$ or $\{\alpha_j : m+1 < j \le n\}$ then (2.12) follows from the relations

$$(x_\alpha^- \otimes t^{\lceil\lambda(h_\alpha)/2\rceil})w_{\omega_m,\lambda+\omega_m} = 0, \quad [x_m^-, x_\alpha^-] = 0.$$

Suppose that $\alpha+\alpha_m\in R^+$ and α is in the span of $\{\alpha_j : 1\le j\le m-1\}$ (resp. $\{\alpha_j : m+1\le j\le n\}$) and we have

$$[x^-_\alpha, x^-_m]=x^-_{\alpha+\alpha_m},\ \ \omega_{m-1}(h_\alpha)=1,\ \ \omega_m(h_\alpha)=\omega_{m+1}(h_\alpha)=0,$$
$$(\text{resp. } [x^-_\alpha, x^-_m]=x^-_{\alpha+\alpha_m},\ \ \omega_{m-1}(h_\alpha)=0=\omega_m(h_\alpha),\ \ \omega_{m+1}(h_\alpha)=1).$$

The defining relations of $M(\omega_m,\lambda+\omega_m)$ give

$$(x^-_\alpha\otimes t^{1+\lceil\lambda(h_\alpha)/2\rceil})w_{\omega_m,\lambda+\omega_m}=0=(x^-_{\alpha+\alpha_m}\otimes t^{(2+\lceil\lambda(h_\alpha)/2\rceil)})w_{\omega_m,\lambda+\omega_m},$$

and (2.12) follows in this case.

Finally we must consider the case when $\alpha=\beta+\alpha_m+\gamma$ with $\omega_{m-1}(h_\beta)=1=\omega_{m+1}(h_\gamma)$. Then x^-_α and x^-_m commute and since

$$\omega_{m-1}(h_\alpha)+\lceil(\lambda+\omega_{m+1})(h_\alpha)/2\rceil=\omega_m(h_\alpha)+\lceil(\lambda+\omega_m)(h_\alpha)/2\rceil$$

we see that (2.12) becomes the relation

$$(x^-_m\otimes t)(x^-_\alpha\otimes t^{\omega_m(h_\alpha)+\lceil(\lambda+\omega_m)(h_\alpha)/2\rceil})w_{\omega_m,\lambda+\omega_m}=0,$$

in $M(\omega_m,\lambda+\omega_m)$. This completes the proof of the existence of the right exact sequence in (ii).

In particular, we have proved that for (ν,λ) compatible

$$\dim M(\nu,\lambda)\le\dim M(\nu-2\omega_j,\lambda+2\omega_j)+\dim M(\nu-\alpha_j,\lambda),\ \ \nu(h_j)\ge 2, \tag{2.13}$$

and if $\lambda=2\lambda_0+\lambda_1$ and $\nu\in P^+(1)$ with $m=\max\nu<\min\lambda_1=p$, then

$$\dim M(\nu,\lambda)\le\dim M(\nu-\omega_m,\lambda+\omega_m)+\dim M(\nu-\omega_m+\omega_{m-1},\lambda-\omega_p+\omega_{p+1}). \tag{2.14}$$

If $\lambda\in P^+(1)$ and $m<\min\lambda$ then

$$\dim M(\omega_m,\lambda+\omega_m)\le\dim M(\omega_{m-1},\lambda+\omega_{m+1})+\dim M(0,\lambda+2\omega_m). \tag{2.15}$$

2.7. To show that the right exact sequences established are exact it suffices to prove that equality holds in (2.13), (2.14) and (2.15). This requires some additional work and we begin with the definition of the fusion product of two $\mathfrak{g}[t]$-modules. Given a module V for $\mathfrak{sl}_{n+1}[t]$ and $z\in\mathbb{C}$ denote by V^z the $\mathfrak{sl}_{n+1}[t]$-module with underlying vector space V and action given by,

$$(x\otimes t^r)w=(x\otimes(t+z)^r)w,\ \ x\in\mathfrak{sl}_{n+1},\ \ r\in\mathbb{Z}_+,\ w\in V.$$

Suppose that V_1, V_2 are cyclic finite–dimensional $\mathfrak{sl}_{n+1}[t]$-modules with cyclic vectors v_1 and v_2 respectively. It was proved in [11] that if z_1, z_2 are distinct complex numbers, then the tensor product $V_1^{z_1} \otimes V_2^{z_2}$ is a cyclic $\mathfrak{sl}_{n+1}[t]$-module generated by $v_1 \otimes v_2$. Further this module admits a filtration by the non–negative integers: the r-th filtered piece of $V_1^{z_1} \otimes V_2^{z_2}$ is spanned by elements of the form $(y_1 \otimes t^{s_1}) \cdots (y_m \otimes t^{s_m})(v_1 \otimes v_2)$ where $m \geq 0$, $y_1, \cdots, y_m \in \mathfrak{sl}_{n+1}$, $s_1, \cdots, s_m \in \mathbb{Z}_+$ and $s_1 + \cdots + s_m \leq r$. The associated graded space is called a fusion product and is denoted $V_1^{z_1} * V_2^{z_2}$. It admits a canonical $\mathfrak{sl}_{n+1}[t]$-module structure and is generated by the image of $v_1 \otimes v_2$ and,

$$\dim (V_1^{z_1} * V_2^{z_2}) = \dim V_1 \dim V_2.$$

Part (i) of the following proposition was proved in [6] while parts (ii) and (iii) were proved in [9] and [24] respectively. Note that the $M(\nu, 0)$ is denoted as $W_{\mathrm{loc}}(\nu)$ and $M(0, \lambda)$ is denoted as $D(2, \lambda)$ in those papers. Part (iv) was proved in [2] and establishes the exactness of the sequence in Proposition 2.5(i)(b) when $\nu = 0$.

Proposition.

(i) For all $\nu_1, \nu_2 \in P^+$ where $\nu = \nu_1 + \nu_2$, we have

$$\dim M(\nu, 0) = \dim M(\nu_1, 0) \dim M(\nu_2, 0).$$

(ii) Suppose that $\lambda, \mu \in P^+$ with $\lambda = 2\lambda_0 + \lambda_1$ are such that $\lambda_0 - \mu \in P^+$. Then

$$\dim M(0, \lambda) = \dim M(0, \lambda - 2\mu) \dim M(0, 2\mu).$$

(iii) For $\lambda, \nu \in P^+$ we have

$$\dim M(\nu, \lambda) \geq \dim M(\nu, 0) \dim M(0, \lambda).$$

(iv) For $m \in [1, n]$ and $\lambda \in P^+(1)$ with $m < \min \lambda = p$, we have

$$\dim M(\omega_m, \lambda) = \dim M(\omega_m, 0) \dim M(0, \lambda)$$

and there exists a short exact sequence of $\mathfrak{g}[t]$-modules

$$0 \to \tau_1^* M(\omega_{m-1}, \lambda - \omega_p + \omega_{p+1}) \to M(\omega_m, \lambda) \to M(0, \lambda + \omega_m) \to 0.$$

Proof. We give an outline of the proof of (iii) given by Wand in [24]. First, the author constructs a surjective map

$$M(\nu, \lambda) \twoheadrightarrow M(\nu, 0) * M(0, \lambda)$$

where $M(\nu,0) * M(0,\lambda)$ is the fusion of the two modules $M(\nu,0)$ and $M(0,\lambda)$ (recall that briefly the fusion product is a graded tensor product of graded modules, originally defined in [10]). Let $v_{\nu,\lambda}$, v_ν, and v_μ be the generators of $M(\nu,\lambda), M(\nu,0)$ and $M(0,\lambda)$ respectively. The surjective map is defined in the natural way by mapping the generator $v_{\nu,\lambda}$ to $v_\nu * v_\mu$. Next, we verify (see [24, Section 3.2, Theorem 14] for details) that the defining relations (2.1) and (2.2) for $M(\nu,\lambda)$ are satisfied by $v_\nu * v_\mu$, which is proven using the definition of the fusion product. Once the surjection is established, the dimension inequality follows immediately. $\square$

2.8. Let $(\nu,\lambda) \in P^+ \times P^+$ be a compatible pair. Then,

- if $\nu(h_j) \geq 2$, for some $j \in [1,n]$ the pairs $(\nu - 2\omega_j, \lambda + 2\omega_j)$ and $(\nu - \alpha_j, \lambda)$ are compatible.
- If $\nu \in P^+(1)$ with $\max \nu < \min \lambda_1 = m$, the pair $(\nu + \omega_m, \lambda - \omega_m)$ is compatible. If in addition $p = \min(\lambda_1 - \omega_m) > 0$, then the pair $(\nu + \omega_{m-1}, \lambda - \alpha_{m,p} - \omega_{m-1})$ is also compatible.

We shall use this observation freely in the rest of the paper. Define a partial order on the set of compatible pairs as follows:

$$(\nu',\lambda') \leq (\nu,\lambda) \text{ if } \nu + \lambda - \nu' - \lambda' \in Q^+ \setminus \{0\} \text{ or } \nu' + \lambda' = \nu + \lambda \text{ and } \nu - \nu' \in P^+.$$

We claim that the elements $\{(0,\omega_i) : i \in [0,n]\}$ are the minimal elements with respect to this order. To see this, suppose that (ν,λ) is compatible. If $\nu(h_j) \geq 2$ for some $j \in [1,n]$ the pair $(\nu - 2\omega_j, \lambda + 2\omega_j)$ is compatible and less than (ν,λ). If $\nu \in P^+(1)$ and $m = \min \nu > 0$ then $(\nu - \omega_m, \lambda + \omega_m)$ is compatible and less than (ν,λ). If $\nu = 0$ and $i \in [0,n]$ is such that $\lambda - \omega_i \in Q^+ \setminus \{0\}$ then $(0,\omega_i) < (0,\lambda)$ and hence the claim is proved.

The following result shows that equality holds in (2.13), (2.14) and (2.15) and completes the proof of Proposition 2.5.

Proposition. *Let (ν,λ) be compatible.*

(i) For all (ν,λ) we have,

$$\dim M(\nu,\lambda) = \dim M(\nu,0) \dim M(0,\lambda).$$

(ii) If $\nu(h_j) \geq 2$, then

$$\dim M(\nu,\lambda) = \dim M(\nu - 2\omega_j, \lambda + 2\omega_j) + \dim M(\nu - \alpha_j, \lambda).$$

(iii) If $\lambda = 2\lambda_0 + \lambda_1$ *and* $\nu \in P^+(1)$ *with* $m = \max \nu < \min \lambda_1 = p$,

$$\dim M(\nu, \lambda) = \dim M(\nu - \omega_m, \lambda + \omega_m) + \dim M(\nu - \omega_m + \omega_{m-1}, \lambda - \omega_p + \omega_{p+1}).$$

(iv) If $\lambda \in P^+(1)$ *and* $m < \min \lambda$ *then*

$$\begin{aligned}\dim M(\omega_m, \lambda + \omega_m) &= \dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m) \\ &= \dim M(\omega_{m-1}, \lambda + \omega_{m+1}) + \dim M(0, \lambda + 2\omega_m).\end{aligned}$$

Proof. The proof of (i)-(iii) proceeds by an induction on the partial order on compatible pairs. Induction begins for the minimal elements, since

$$\dim M(0, \omega_i) = \dim M(\omega_i, 0) = \dim V(\omega_i), \quad i \in [0, n].$$

Before proving the inductive step we note the following. Using Proposition 2.7(i) and the character theory of $\mathfrak{sl}_n$, we get for all $j \in [1, n]$,

$$\begin{aligned}\dim M(2\omega_j, 0) = (\dim V(\omega_j))^2 &= \dim V(2\omega_j) + \dim V(\omega_{j-1}) \dim V(\omega_{j+1}) \\ &= \dim M(0, 2\omega_j) + \dim M(2\omega_j - \alpha_j, 0)\end{aligned}$$

Assume that the result holds for all compatible pairs $(\nu', \lambda') < (\nu, \lambda)$. Suppose that there exists $j \in [1, n]$ with $\nu(h_j) \geq 2$. Then Proposition 2.7(iii) and (2.13) give

$$\dim M(\nu, 0) \dim M(0, \lambda) \leq \dim M(\nu, \lambda) \leq \dim M(\nu - 2\omega_j, \lambda + 2\omega_j) + \dim M(\nu - \alpha_j, \lambda).$$

Using the inductive hypothesis and part (ii) of Proposition 2.7 we get,

$$\begin{aligned}&\dim M(\nu - 2\omega_j, \lambda + 2\omega_j) + \dim M(\nu - \alpha_j, \lambda) \\ &= \dim M(\nu - 2\omega_j, 0) \dim M(0, \lambda + 2\omega_j) + \dim M(\nu - \alpha_j, 0) \dim M(0, \lambda) \\ &= (\dim M(\nu - 2\omega_j, 0) \dim M(0, 2\omega_j) + \dim M(\nu - \alpha_j, 0)) \dim M(0, \lambda) \\ &= \dim M(\nu, 0) \dim M(0, \lambda),\end{aligned}$$

as needed.

Suppose that $\nu \in P^+(1)$ with $m = \min \nu$. If $\lambda = 2\lambda_0$, the isomorphism in (2.4) gives

$$\dim M(\nu, 2\lambda_0) = \dim M(\nu - \omega_m, 2\lambda_0 + \omega_m).$$

Since $(\nu - \omega_m, \omega_m + 2\lambda_0) < (\nu, 2\lambda_0)$ the inductive hypothesis, equation (2.4) and Proposition 2.7(ii) give

$$\begin{aligned}&\dim M(\nu - \omega_m, 2\lambda_0 + \omega_m) = \dim M(\nu - \omega_m, 0) \dim M(0, \omega_m + 2\lambda_0) \\ &= \dim M(\nu - \omega_m, 0) \dim M(\omega_m, 0) \dim M(0, 2\lambda_0) = \dim M(\nu, 0) \dim M(0, 2\lambda_0),\end{aligned}$$

where the last equality follows from Proposition 2.7(i).

If $\nu \in P^+(1)$ with $\min \nu = m$ and $p = \min \lambda_1 > 0$ Proposition 2.7(iii) and (2.14) give

$$\dim M(\nu, 0) \dim M(0, \lambda) \le \dim M(\nu, \lambda) \le$$
$$\dim M(\nu - \omega_m, \lambda + \omega_m) + \dim M(\nu - \omega_m + \omega_{m-1}, \lambda - \omega_p + \omega_{p+1}).$$

The inductive hypothesis applies to the modules on the right hand side of the preceding inequality and so we get

$$\begin{aligned}\dim M(\nu - \omega_m, \lambda + \omega_m) &= \dim M(\nu - \omega_m, 0) \dim M(0, \lambda + \omega_m)\\ &= \dim M(\nu - \omega_m, 0)\big(\dim M(\omega_m, 0) \dim M(0, \lambda)\\ &\quad - \dim M(\omega_{m-1}, 0) \dim M(0, \lambda - \omega_p + \omega_{p+1})\big)\\ &= \dim M(\nu, 0) \dim M(0, \lambda) - \dim M(\nu - \omega_m + \omega_{m-1}, \lambda - \omega_p + \omega_{p+1})\end{aligned}$$

where the second equality follows from Proposition 2.7(ii) and (iv). For the last equality we use Proposition 2.7(i) and the inductive hypothesis. This completes the proof of part (i)-(iii) of this proposition.

The proof of part (iv) is similar with (2.15) showing that it is enough to prove

$$\dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m) = \dim M(\omega_{m-1}, \lambda + \omega_{m+1}) + \dim M(0, \lambda + 2\omega_m).$$

We proceed by a downward induction on m. If $m = n$ then $\lambda = 0$ by our assumptions and the result is known from our initial observations since $M(\omega_m, \omega_m) \cong M(2\omega_m, 0)$. For the inductive step, Proposition 2.7(iii) and (2.15) give

$$\begin{aligned}\dim M(\omega_m, 0) \dim M(0, \lambda + \omega_m) &\le \dim M(\omega_m, \lambda + \omega_m)\\ &\le \dim M(\omega_{m-1}, \lambda + \omega_{m+1}) + \dim M(0, \lambda + 2\omega_m)\end{aligned}$$

Using Proposition 2.7(ii), we have

$$\dim M(0, 2\omega_m + \lambda) = (\dim M(2\omega_m, 0) - \dim M(\omega_{m-1}, 0) \dim M(\omega_{m+1}, 0)) \dim M(0, \lambda). \tag{2.16}$$

If we let $p = \min \lambda_1$ then (ω_{m+1}, λ) is compatible if $p > m + 1$ and part (i) of this proposition applies; if $p = m + 1$ the inductive hypothesis on m applies to (ω_{m+1}, λ). Hence

$$\dim M(\omega_{m+1}, 0) \dim M(0, \lambda) = \dim M(0, \lambda + \omega_{m+1}) + \dim M(\omega_m, \lambda - \omega_p + \omega_{p+1}).$$

Multiplying through by $\dim M(\omega_{m-1}, 0)$ and using the fact that $(\omega_{m-1}, \lambda + \omega_{m+1})$ is compatible we have

$$\dim M(\omega_{m-1},0)\dim M(\omega_{m+1},0)\dim M(0,\lambda) =$$
$$\dim M(\omega_{m-1},\lambda+\omega_{m+1}) + \dim M(\omega_{m-1},0)\dim M(\omega_m,\lambda-\omega_p+\omega_{p+1})$$

Substituting in (2.16) and using the fact that $(\omega_m, \lambda-\omega_p+\omega_{p+1})$ is compatible gives

$$\begin{aligned}\dim M(\omega_{m-1},\lambda+\omega_{m+1}) &+ \dim M(0,\lambda+2\omega_m)\\ &= \dim M(\omega_m,0)\big(\dim M(\omega_m,0)\dim M(0,\lambda)\\ &- \dim M(\omega_{m-1},0)\dim M(0,\lambda-\omega_p+\omega_{p+1})\big)\\ &= \dim M(\omega_m,0)\dim M(0,\lambda+\omega_m)\end{aligned}$$

where the last equality follows since (ω_m, λ) is compatible and we can use part (i) of this proposition. □

The following corollary proves the second assertion of Theorem 2.

Corollary. *For (ν, λ) compatible, we have an isomorphism of $\mathfrak{g}[t]$–modules*

$$M(\nu,\lambda) \cong M(\nu,0) * M(0,\lambda).$$

Proof. Recall that we proved in Proposition 2.7(ii) that there exists a surjective map of $\mathfrak{g}[t]$–modules

$$M(\nu,\lambda) \twoheadrightarrow M(\nu,0) * M(0,\lambda).$$

The corollary is immediate from part (i) of Proposition 2.8. □

3. The elements $g^{\mu}_{\nu,\lambda}$

We analyze recursive formulae for $g^{\mu}_{\nu,\lambda}$ which follow from Proposition 2.5 and establish (see Section 3.2) Proposition 2.3 for the elements $g^{\mu}_{\nu,\lambda}$. We also give closed formulae for the $g^{\mu}_{\nu,\lambda}$ and establish a technical result which will be needed in the study of the polynomials $h^{\mu}_{\nu,\lambda}$.

3.1. The next result is an immediate consequence of Proposition 2.5

Lemma. *Assume that (ν, λ) is compatible. If $\nu(h_j) \geq 2$ for some $j \in [1, n]$ we have for $b \in \{g, h\}$*

$$b^{\mu}_{\nu,\lambda} = b^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\lambda_0+\nu,\alpha_j)-1} b^{\mu}_{\nu-\alpha_j,\lambda},$$

and if $\nu \in P^+(1)$ with $\max \nu < \min \lambda_1 = m$ and $p = \min(\lambda_1 - \omega_m) > 0$,

$$b^{\mu}_{\nu+\omega_m,\lambda-\omega_m} = b^{\mu}_{\nu,\lambda} + q^{\frac{1}{2}(\lambda,\alpha_{m,p})} b_{\nu+\omega_{m-1},\lambda-\alpha_{m,p}-\omega_{m-1}}.$$

Finally if $\lambda \in P^+(1)$ *and* $m < \min\lambda$ *we have*

$$b^{\mu}_{\omega_m,\lambda+\omega_m} = b^{\mu}_{0,\lambda+2\omega_m} + q b^{\mu}_{\omega_{m-1},\lambda+\omega_{m+1}}. \tag{3.1}$$

Proof. If $\nu(h_j) \geq 2$, then Proposition 2.5(i)(a) gives an equality of graded characters

$$\mathrm{ch}_{\mathrm{gr}}\, M(\nu,\lambda) = \mathrm{ch}_{\mathrm{gr}}\, M(\nu - 2\omega_j, \lambda + 2\omega_j) + q^{(\nu+\lambda_0,\alpha_j)-1}\, \mathrm{ch}_{\mathrm{gr}}\, M(\nu - \alpha_j, \lambda).$$

Using (2.5) and equating coefficients of $M(\mu, 0)$ (resp. $M(0,\mu)$) on both sides gives the first assertion of the Lemma when $b = g$ (resp. $b = h$). The proof of the other assertions is identical. □

3.2. We show by induction on $\mathrm{ht}\,\lambda$ that if (ν,λ) is compatible, then

$$g^{\mu}_{\nu,\lambda} = q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}.$$

If $\mathrm{ht}\,\lambda = 0$ this follows since $g^{\mu}_{\nu,0} = \delta_{\nu,\mu} = g^{\mu-\nu}_{0,0}$. For the inductive step we consider various cases.

Suppose first that $\lambda_1 = 0$ and let $j \in [1,n]$ be such that $(\lambda_0,\alpha_j) > 0$. Then $(\nu + 2\omega_j, \lambda - 2\omega_j)$ is compatible and we have,

$$\begin{aligned} g^{\mu}_{\nu,\lambda} &= g^{\mu}_{\nu+2\omega_j,\lambda-2\omega_j} - q^{(\lambda_0+\omega_j+\nu,\alpha_j)-1} g^{\mu}_{\nu+2\omega_j-\alpha_j,\lambda-2\omega_j} \\ &= q^{(\lambda+\nu-\mu,\nu+2\omega_j)} g^{\mu-\nu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0+\omega_j+\nu,\alpha_j)-1+(\lambda+\nu-\alpha_j-\mu,\nu+2\omega_j-\alpha_j)} g^{\mu-\nu-2\omega_j+\alpha_j}_{0,\lambda-2\omega_j} \\ &= q^{(\lambda+\nu-\mu,\nu)} \left(g^{\mu-\nu}_{2\omega_j,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)} g^{\mu-\nu}_{\omega_{j-1}+\omega_{j+1},\lambda-2\omega_j} \right) \\ &= q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{0,\lambda}. \end{aligned}$$

Here the first equality follows from Lemma 3.1, the second and third follow from the inductive hypothesis applied to the compatible pairs $(\nu+2\omega_j, \lambda-2\omega_j)$, $(\nu+2\omega_j-\alpha_j, \lambda-2\omega_j)$ and $(2\omega_j, \lambda-2\omega_j)$, $(2\omega_j-\alpha_j, \lambda-2\omega_j)$ respectively. The fourth equality is a further application of Lemma 3.1.

If $\mathrm{ht}\,\lambda_1 = 1$ and $m = \min\lambda_1$ then $(\nu+\omega_m, \lambda-\omega_m)$ is compatible. Since $\lambda - \omega_m = 2\lambda_0$ the isomorphism in (2.4) and the inductive hypothesis gives

$$\begin{aligned} g^{\mu}_{\nu,\omega_m+2\lambda_0} &= g^{\mu}_{\nu+\omega_m,2\lambda_0} = q^{(\lambda+\nu-\mu,\nu+\omega_m)} g^{\mu-\omega_m-\nu}_{0,2\lambda_0} = q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{\omega_m,2\lambda_0} \\ &= q^{(\lambda+\nu-\mu,\nu)} g^{\mu-\nu}_{0,\omega_m+2\lambda_0}. \end{aligned}$$

For $\mathrm{ht}\,\lambda_1 = r \geq 2$, let $m = \min\lambda_1$ and $p = \min(\lambda_1 - \omega_m)$. Since $(\nu+\omega_m, \lambda-\omega_m)$ is compatible, Lemma 3.1 gives

$$g^{\mu}_{\nu,\lambda} = g^{\mu}_{\nu+\omega_m,\lambda-\omega_m} - q^{(\nu+\frac{1}{2}\lambda,\alpha_{m,p})} g^{\mu}_{\nu+\omega_{m-1},\lambda-\omega_m-\omega_p+\omega_{m+1}}.$$

The inductive hypothesis applies to both terms on the right hand side and so

$$\begin{aligned} g^{\mu}_{\nu,\lambda} &= q^{(\lambda+\nu-\mu,\nu+\omega_m)} g^{\mu-\nu-\omega_m}_{0,\lambda-\omega_m} - q^{(\nu+\frac{1}{2}\lambda,\alpha_{m,p})+(\lambda+\nu-\alpha_{m,p}-\mu,\nu+\omega_{m-1})} g^{\mu-\nu-\omega_{m-1}}_{0,\lambda-\omega_m-\omega_p+\omega_{m+1}}, \\ &= q^{(\lambda+\nu-\mu,\nu)} \left(g^{\mu-\nu}_{\omega_m,\lambda-\omega_m} - q^{\frac{1}{2}(\lambda,\alpha_{m,p})} g^{\mu-\nu}_{\omega_{m-1},\lambda-\omega_m-\omega_p+\omega_{p+1}} \right), \\ &= g^{\mu-\nu}_{0,\lambda}, \end{aligned}$$

where the second equality is again an application of the inductive hypothesis and the final equality follows from a further application of Lemma 3.1.

3.3. We establish some further properties of the polynomials $g^{\mu}_{0,\lambda}$.

Proposition. *For $\lambda \in P^+$ with $\lambda = 2\lambda_0 + \lambda_1$ and $m = \min \lambda_1$, $p = \min(\lambda_1 - \omega_m)$, we have,*

$$\sum_{\mu \in P} q^{\frac{1}{2}(\mu,\mu)} g^{\mu}_{0,\lambda} = 0, \tag{3.2a}$$

$$g^{\mu}_{0,\lambda} \neq 0 \implies (\lambda - \mu, \omega_s) \leq (\lambda, \alpha_s) \ \textit{for all} \ 1 \leq s < m, \tag{3.2b}$$

$$g^{\mu}_{0,\lambda} = g^{\mu-\omega_m}_{0,\lambda-\omega_m} - q(1-\delta_{p,0}) g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}, \quad \lambda_0 = 0, \tag{3.2c}$$

$$g^{\mu}_{0,\lambda} = g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q g^{\mu-2\omega_j+\alpha_j}_{0,\lambda-2\omega_j}, \quad \lambda_0 = \omega_j \ \textit{and} \ j+1 < m. \tag{3.2d}$$

Proof. Recall that the pairs $(\omega_m, \lambda - \omega_m)$ and $(\omega_{m-1}, \lambda - \alpha_{m,p} - \omega_{m-1})$ are compatible if $m > 0$ and also that if $(\lambda_0, \alpha_j) > 0$ for some $j \in [1, n]$ then the pairs $(2\omega_j, \lambda - 2\omega_j)$ and $(2\omega_j - \alpha_j, \lambda - 2\omega_j)$ are compatible. Hence Lemma 3.1 and the results of Section 3.2 give

$$g^{\mu}_{0,\lambda} = q^{(\omega_m,\lambda-\mu)} g^{\mu-\omega_m}_{0,\lambda-\omega_m} - (1-\delta_{p,0}) q^{\frac{1}{2}(\lambda,\alpha_{m,p})+(\omega_{m-1},\lambda-\mu)} g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}}, \quad m > 0, \tag{3.3}$$

$$g^{\mu}_{0,\lambda} = q^{(2\omega_j,\lambda-\mu)} g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} - q^{(\lambda_0,\alpha_j)+(2\omega_j-\alpha_j,\lambda-\alpha_j-\mu)} g^{\mu-2\omega_j+\alpha_j}_{0,\lambda-2\omega_j}. \tag{3.4}$$

An obvious induction on $\operatorname{ht}\lambda$ proves (3.2a), where we use (3.3) if $m > 0$ and (3.4) if $m = 0$.

We prove (3.2b)-(3.2d) and notice that in all cases $m > 0$. If $\operatorname{ht}\lambda = 0$, then $\mu = 0$ and the statement is obviously true. Suppose that (3.2b) is true for all $\nu \in P^+$ with $\operatorname{ht}\nu < M$. We prove that the inductive step holds when $\lambda \in P^+(1)$ with $\operatorname{ht}\lambda = M$. Note that $g^{\mu}_{0,\lambda} \neq 0$ only if one of the terms on the right hand side of (3.3) is nonzero. Since $\operatorname{ht}(\lambda - \omega_m) = M - 1 = \operatorname{ht}(\lambda - \alpha_{m,p} - \omega_{m-1})$ we see that

$$g^{\mu-\omega_m}_{0,\lambda-\omega_m} \neq 0 \implies (\lambda-\mu,\omega_s) \le (\lambda-\omega_m,\alpha_s) \le (\lambda,\alpha_s) \quad \text{for all } 1 \le s < p,$$
$$g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}\omega_{m-1}} \neq 0 \implies (\lambda-\mu,\omega_s) \le (\lambda,\alpha_s) - (\omega_{m-1},\alpha_s) - (\alpha_{m,p},\alpha_s-\omega_s),$$
$$1 \le s < r,$$

where $r = \min(\lambda - \omega_{m-1} - \alpha_{m,p})$. Notice that $r > p > m$. Hence if $g^{\mu-\omega_m}_{0,\lambda-\omega_m} \neq 0$ it is clear that (3.2b) holds for $g^{\mu}_{0,\lambda}$. If $g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}\omega_{m-1}} \neq 0$ then observing that

$$(\lambda,\alpha_s) - (\omega_{m-1},\alpha_s) - (\alpha_{m,p},\alpha_s-\omega_s) = (\lambda,\alpha_s) \;\; 1 \le s < m,$$

we see that the inductive step is again proved. If $\lambda \notin P^+(1)$ the inductive step is proved by using (3.4) in a similar manner. To establish (3.2c) it is enough to note that (3.2b) applied to $(\lambda-\omega_m, \mu-\omega_m)$ and $(\lambda-\alpha_{m,p}-\omega_{m-1}, \mu-\omega_{m-1})$ gives

$$g^{\mu-\omega_m}_{0,\lambda-\omega_m} \neq 0 \implies (\lambda-\mu,\omega_m) = 0, \;\; g^{\mu-\omega_{m-1}}_{0,\lambda-\alpha_{m,p}-\omega_{m-1}} \neq 0 \implies (\lambda-\mu-\alpha_{m,p},\omega_{m-1}) = 0.$$

To establish (3.2d) recall that $\lambda_0 = \omega_j$ and $j+1 < m$. This means that $(\lambda-2\omega_j,\alpha_j) = 0$. Since $\lambda-\mu \in Q^+$ when $g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} \neq 0$, we get by applying (3.2b) to $(\lambda-2\omega_j, \mu-2\omega_j)$, that

$$g^{\mu-2\omega_j}_{0,\lambda-2\omega_j} \neq 0 \implies 0 \le (\lambda-\mu,\omega_j) \le (\alpha_j,\lambda-2\omega_j) = 0.$$

Next, applying (3.2b) to $(\lambda-2\omega_j, \mu-2\omega_j+\alpha_j)$ in the two cases that $s = j-1$ and $s = j+1$ and noting that $\lambda-\mu-\alpha_j \in Q^+$ when $g^{\mu-2\omega_j+\alpha_j}_{\lambda-2\omega_j} \neq 0$ a similar argument gives

$$g^{\mu-2\omega_j+\alpha_j}_{\lambda-2\omega_j} \neq 0 \implies (\omega_{j-1},\lambda-\alpha_j-\mu) = 0, \;\; (\omega_{j+1},\lambda-\alpha_j-\mu) = 0.$$

Hence, $(2\omega_j-\alpha_j, \lambda-\alpha_j-\mu) = 0$. □

3.4. We shall need the following result in the next section.

Proposition. *For* $m \in [1,n]$ *and* $\lambda \in P^+(1)$ *with* $m < p = \min\lambda$ *we have*

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}} = q^{(\lambda+2\omega_m-\mu,\ \omega_m)} g^{\mu-\omega_m}_{0,\lambda+\omega_m}, \tag{3.5}$$

Proof. We prove the first equality. For this, we note that the third equality in Lemma 3.1 gives

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q g^{\mu}_{\omega_{m-1},\lambda+\omega_{m+1}}.$$

Using Proposition 2.3 on the compatible pair $(\omega_{m-1}, \lambda+\omega_{m+1})$ shows that the previous equation can be rewritten as,

$$g^{\mu}_{\omega_m,\lambda+\omega_m} = g^{\mu}_{0,\lambda+2\omega_m} + q^{1+(\omega_{m-1},\ \lambda+2\omega_m-\alpha_m-\mu)} g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}}.$$

Hence it suffices to show that $(\omega_{m-1}, \lambda + 2\omega_m - \alpha_m - \mu) = 0$. For this we recall that $g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}} = 0$ unless $\lambda - \mu + \omega_{m+1} + \omega_{m-1} \in Q^+$. This forces

$$(\omega_{m-1}, \lambda + 2\omega_m - \alpha_m - \mu) \geq 0.$$

On the other hand, using (3.2b) (with $s = m - 1$) we see that $g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}} \neq 0$ gives

$$0 \leq (\lambda + \omega_{m-1} - \mu + \omega_{m+1}, \omega_{m-1}) \leq (\lambda + \omega_{m+1}, \alpha_{m-1}) \leq 0$$

where the last inequality hold since, $\min \lambda > m$ and the proof of the first equality is complete.

We prove the second equality of (3.5) by a downward induction on m. If $m = n$ then $\lambda = 0$ and

$$g^{\mu-\omega_m}_{0,\ \omega_m} = \delta_{\mu-\omega_m,\omega_m}, \quad g^{\mu-\omega_{m-1}}_{0,0} = \delta_{\mu-\omega_{m-1},0}.$$

Hence when $m = n$ the second equality of (3.5) follows if

$$\delta_{\mu,2\omega_n} = g^{\mu}_{0,2\omega_n} + q\delta_{\mu,\omega_{n-1}}.$$

However, this is precisely (3.4) applied to $2\omega_n$. For the inductive step we use (3.2c) applied to the compatible pair (ω_m, λ) and either (3.2c) or the inductive hypothesis to (ω_{m+1}, λ) to get

$$g^{\mu-\omega_m}_{0,\lambda+\omega_m} = g^{\mu-2\omega_m}_{0,\lambda} - qg^{\mu-\omega_m-\omega_{m-1}}_{0,\lambda-\omega_p+\omega_{p+1}},$$
$$g^{\mu-\omega_{m-1}}_{0,\lambda+\omega_{m+1}} = g^{\mu-2\omega_m+\alpha_m}_{0,\lambda} - qg^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}},$$

and hence we must prove that

$$g^{\mu}_{0,\lambda+2\omega_m} = \left(q^{(\lambda+2\omega_m-\mu,\ \omega_m)} g^{\mu-2\omega_m}_{0,\lambda} - qg^{\mu-2\omega_m+\alpha_m}_{0,\lambda}\right)$$
$$-q(q^{(\lambda+2\omega_m-\mu,\ \omega_m)} - q) g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}}.$$

Using (3.2b) we see that

$$g^{\mu-\omega_{m-1}-\omega_m}_{0,\lambda-\omega_p+\omega_{p+1}} \neq 0 \implies (\lambda - \mu + 2\omega_m - \alpha_{m,p}, \omega_m) = 0 \implies (\lambda + 2\omega_m - \mu, \omega_m) = 1,$$
$$g^{\mu-2\omega_m}_{0,\lambda} \neq 0 \implies (\lambda + 2\omega_m - \mu, \omega_m) = 0,$$

and hence it suffices to prove that

$$g^{\mu}_{0,\lambda+2\omega_m} = g^{\mu-2\omega_m}_{0,\lambda} - qg^{\mu-2\omega_m+\alpha_m}_{0,\lambda}.$$

If $p > m+1$ the pair $(2\omega_m, \lambda)$ is compatible and the preceding equation is precisely (3.4). Suppose that $p = m+1$ and $\lambda = \omega_{m+1}$ we have

$$q^{(\omega_{m+1},\omega_{m+1}+2\omega_m-\mu)} g^{\mu-\omega_{m+1}}_{0,2\omega_m} = g^{\mu}_{\omega_{m+1},2\omega_m} = g^{\mu}_{0,\omega_{m+1}+2\omega_m}.$$

Hence we must prove that

$$\delta_{\omega_{m+1},\mu-2\omega_m} = \delta_{\mu-\omega_{m+1},2\omega_m} - q\delta_{\mu-\omega_{m+1},2\omega_m-\alpha_m} + q\delta_{\omega_{m+1},\mu-2\omega_m+\alpha_m},$$

and this is clear. Finally suppose that $p = m+1$ and $r = \min(\lambda - \omega_p) > 0$. Using the fact that $(\omega_{m+1}, \lambda - \omega_{m+1})$, $(2\omega_m, \lambda - \omega_{m+1})$ and $(2\omega_m, \lambda - \omega_{m+1} - \omega_r + \omega_{r+1})$ are compatible, and using Lemma 1 and Section 3.2 gives the following equalities:

$$\begin{aligned} g^{\mu-2\omega_m}_{0,\lambda} &= g^{\mu-2\omega_m-\omega_{m+1}}_{0,\lambda-\omega_{m+1}} - qg^{\mu-3\omega_m}_{0,\lambda-\omega_{m+1}-\omega_r+\omega_{r+1}} = g^{\mu-\omega_{m+1}}_{2\omega_m,\lambda-\omega_{m+1}} - qg^{\mu-\omega_m}_{2\omega_m,\lambda-\omega_m-\alpha_{m+1,r}}, \\ &= \left(g^{\mu-\omega_{m+1}}_{0,\lambda+2\omega_m-\omega_{m+1}} + qg^{\mu-2\omega_{m+1}-\omega_{m-1}}_{0,\lambda-\omega_{m+1}}\right) - \left(g^{\mu-\omega_m}_{0,\lambda+\omega_m-\alpha_{m+1,r}} + qg^{\mu-\omega_m-\omega_{m-1}-\omega_{m+1}}_{0,\lambda-\omega_m-\alpha_{m+1,r}}\right) \\ &= \left(g^{\mu-\omega_{m+1}}_{0,\lambda+2\omega_m-\omega_{m+1}} - qg^{\mu-\omega_m}_{0,\lambda+\omega_m-\alpha_{m+1,r}}\right) + q\left(g^{\mu-\omega_{m+1}-\omega_{m-1}}_{\omega_{m+1},\lambda-\omega_{m+1}} - qg^{\mu-\omega_{m-1}-\omega_{m+1}}_{\omega_m,\lambda-\omega_m-\alpha_{m+1,r}}\right) \end{aligned}$$

A further application of (3.2d) to the first term and (3.2c) to the second term show that the right hand side of the last equality is precisely the middle of (3.5) and the inductive step is established and the proof of the proposition is complete. □

3.5. *A closed form for* $g^{\mu}_{0,\lambda}$

We now give a closed form for the elements $g^{\mu}_{0,\lambda}$. Notice that these elements are defined completely by the recursive formulae in (3.3) and (3.4) with the initial condition $g^{\mu}_{0,0} = \delta_{\mu,0}$. Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ and for $\eta = \sum_{i=1}^n k_i\alpha_i \in Q^+$ set $\eta^\diamond = \sum_{i=1}^n k_i\omega_i$. A simple checking using the q-binomial identity

$$\begin{bmatrix} m \\ s \end{bmatrix} = q^s \begin{bmatrix} m-1 \\ s \end{bmatrix} + \begin{bmatrix} m-1 \\ s-1 \end{bmatrix}, \quad m, s \in \mathbb{Z}_+$$

now shows that equation (3.4) forces,

$$g^{\mu}_{0,2\lambda_0} = (-1)^{(2\lambda_0-\mu,\rho)} q^{\frac{1}{2}(2\lambda_0-\mu,\ \mu+\rho+(2\lambda_0-\mu)^\diamond)} \prod_{i=1}^n \begin{bmatrix} (\lambda_0,\alpha_i) \\ (2\lambda_0-\mu,\omega_i) \end{bmatrix}. \tag{3.6}$$

Equation (3.2c) now gives a closed form for $g^{\mu}_{0,\lambda}$ when $\operatorname{ht}\lambda_1 = 1$. To give a closed form for $g^{\mu}_{0,\lambda}$ when $\operatorname{ht}\lambda_1 \geq 2$ needs some additional work.

Given any subset S of P^+ and $\lambda \in P^+$, let $\lambda + S = \{\lambda + \mu : \mu \in S\}$. For $s \geq 0$ and $\lambda \in P^+(1)$ define subsets $\Sigma_s(\lambda) = \Sigma^0_s(\lambda) \cup \Sigma^1_s(\lambda)$ of P^+ recursively with initial conditions

$$\Sigma^0_0(\lambda) = \lambda, \quad \Sigma^1_0(\lambda) = \emptyset, \quad \Sigma^0_s(\omega_m) = \Sigma^1_s(\omega_m) = \emptyset, \quad m \in [0,n], \ s > 0.$$

If $s > 0$ and $\operatorname{ht}\lambda \geq 2$ with $m = \min\lambda$, $p = \min(\lambda - \omega_m)$, then we take

$$\Sigma^0_s(\lambda) = \omega_m + \Sigma_s(\lambda - \omega_m)$$

and we take

$$\Sigma^1_s(\lambda) = \begin{cases} \Sigma^0_{s-1}(\lambda - \alpha_{m,p}), & \lambda(h_{p+1}) = 0 \\ \{\lambda - \alpha_{m,p}\}, & \lambda(h_{p+1}) = 1,\ s = 1. \end{cases}$$

Lastly, if $s \geq 2$ and $\lambda(h_{p+1}) = 1$, then we take

$$\Sigma^1_s(\lambda) = (2\omega_{p+1} + \Sigma^0_{s-1}(\lambda - 2\omega_{p+1} - \alpha_{m,p})) \cup (2\omega_{p+1} - \alpha_{p+1} + \Sigma^0_{s-2}(\lambda - 2\omega_{p+1} - \alpha_{m,p})).$$

Here we understand that if $\Sigma^r_s(\lambda) = \emptyset$ for $r \in \{0, 1\}$ then so is the set $\nu + \Sigma^r_s(\lambda)$.

We give a few examples of the sets $\Sigma_s(\lambda)$. For $m < p$,

$$\Sigma^0_s(\omega_m + \omega_p) = \begin{cases} \{\omega_m + \omega_p\}, & s = 0, \\ \emptyset, & s > 0, \end{cases} \qquad \Sigma^1_s(\omega_m + \omega_p) = \begin{cases} \{\omega_{m-1} + \omega_{p+1}\}, & s = 1, \\ \emptyset, & s \neq 1. \end{cases}$$

For $m < p < r$,

$$\Sigma^0_s(\omega_m + \omega_p + \omega_r) = \begin{cases} \{\omega_m + \omega_p + \omega_r,\} & s = 0, \\ \{\omega_m + \omega_{p-1} + \omega_{r+1}\}, & s = 1, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$\Sigma^1_s(\omega_m + \omega_p + \omega_r) = \begin{cases} \{\omega_{m-1} + \omega_{p+1} + \omega_r\}, & s = 1, \\ \{\omega_{m-1} + \omega_p + \omega_{r+1}\}, & s = 2, \\ \emptyset & \text{otherwise}. \end{cases}$$

Our final example is $\lambda = \omega_m + \omega_p + \omega_\ell + \omega_r$ with $m < p < \ell < r$. This is the first time we see the dependence on $\lambda(h_{p+1})$ and the non-empty sets are listed below:

$$\Sigma^0_s(\lambda) = \begin{cases} \{\omega_m + \omega_p + \omega_\ell + \omega_r\}, & s = 0, \\ \{\omega_m + \omega_{p-1} + \omega_{\ell+1} + \omega_r,\ \ \omega_m + \omega_p + \omega_{\ell-1} + \omega_{r+1}\}, & s = 1, \\ \{\omega_m + \omega_{p-1} + \omega_\ell + \omega_{r+1}\}, & s = 2, \end{cases}$$

$$\Sigma^1_s(\lambda) = \begin{cases} \{\omega_{m-1} + \omega_{p+1} + \omega_\ell + \omega_r\}, & s = 1, \\ \{\omega_{m-1} + \omega_p + \omega_{\ell+1} + \omega_r,\ \omega_{m-1} + \omega_{p+1} + \omega_{\ell-1} + \omega_{r+1}\} & s = 2, \ell \neq p+1 \\ \{\omega_{m-1} + \omega_p + \omega_{\ell+1} + \omega_r\} & s = 2 \ \text{ and } \ \ell = p+1, \\ \{\omega_{m-1} + \omega_p + \omega_\ell + \omega_{r+1}\}, & s = 3 \ \text{ and } \ \ell \neq p+1. \end{cases}$$

Lemma. *If $\lambda \in P^+(1)$ and $\mu \in \Sigma_s(\lambda)$ for some $s \geq 0$, then*

$$\mu(h_k) = 0 \text{ for } k < \min\lambda - 1, \quad \mu(h_{\min\lambda-1}) \leq 1 \text{ and } \lambda - \mu \in \sum_{k \geq \min\lambda} \mathbb{Z}_+\alpha_k. \tag{3.7}$$

In particular the sets $\Sigma_s^r(\lambda)$ and $\Sigma_{s'}^{r'}(\lambda)$ are disjoint unless $(s,r) = (s',r')$.

Proof. The proof of the displayed statements is immediate from the definition of $\Sigma_s(\lambda)$ and a straightforward induction on $\operatorname{ht}\lambda$. The same induction also shows that the sets $\Sigma_s^r(\lambda)$ and $\Sigma_{s'}^r(\lambda)$ are disjoint for $r \in \{0,1\}$. It remains to prove that $\Sigma_s^0(\lambda) \cap \Sigma_s^1(\lambda) = \emptyset$. Assume for a contradiction that μ is an element of the intersection. If $m = \min\lambda$ then since $\mu \in \Sigma_s^0(\lambda)$ we can write $\mu = \omega_m + \nu$ and so by (3.7) we have $\nu(h_k) = 0$ for all $k < \min(\lambda - \omega_m) - 1 \leq m$ and so $\mu(h_{m-1}) = 0$. On the other hand since $\mu \in \Sigma_{s'}^1(\lambda)$ we also have that μ is in one of the following sets:

$$\Sigma_{s'-1}^0(\lambda - \alpha_{m,p}), \quad \{\lambda - \alpha_{m,p}\},$$
$$(2\omega_{p+1} + \Sigma_{s-1}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})), \text{ or}$$
$$(2\omega_{p+1} - \alpha_{p+1} + \Sigma_{s-2}^0(\lambda - 2\omega_{p+1} - \alpha_{m,p})).$$

The elements of these sets are non-zero on h_{m-1}, contradicting the fact that $\mu(h_{m-1}) = 0$. $\square$

For $\lambda, \mu \in P^+$, with $\lambda = 2\lambda_0 + \lambda_1$ and $\lambda_1 \neq 0$ we have

$$g_{0,\lambda}^{\mu} = q^{\frac{1}{2}(\lambda_1,\lambda)} \sum_{s \geq 0} (-1)^s \sum_{\nu \in \Sigma_s(\lambda_1)} q^{\frac{1}{2}(2\lambda_0 - 2\mu + \nu, \nu)} g_{0,2\lambda_0}^{\mu-\nu}, \tag{3.8}$$

where we understand that the second summation is zero if $\Sigma_s(\lambda_1) = \emptyset$. The proof is a tedious calculation which amounts to establishing that the right hand side of (3.8) satisfies equation (3.3).

4. The polynomials $h_{\nu,0}^{\mu}(q)$

4.1. Recall from Section 1 that for $\nu, \mu \in P^+$ with $\mu = 2\mu_0 + \mu_1$, we defined

$$p_\nu^\mu = q^{\frac{1}{2}(\nu+\mu_1,\ \nu-\mu)} \prod_{j=1}^{n} \begin{bmatrix} (\nu-\mu,\ \omega_j) + (\mu_0, \alpha_j) \\ (\nu-\mu, \omega_j) \end{bmatrix}$$

and $p_\nu^\mu = 0$ if ν or μ is not in P^+. It is easy to check that

$$p_\nu^\mu = q^{(\omega_j,\nu-\mu)} p_{\nu-2\omega_j}^{\mu-2\omega_j} + q^{(\nu,\alpha_j)-1} p_{\nu-\alpha_j}^{\mu} \qquad j \in [1,n], \quad \nu(h_j) \geq 2,$$
$$p_{\nu+\omega_m}^{\mu+\omega_m} = q^{(\nu-\mu,\omega_m)} p_\nu^\mu, \quad \text{if } \mu(h_m) \in 2\mathbb{Z}_+.$$

In the rest of this section we will prove that

$$h^{\mu}_{\nu,0} = p^{\mu}_{\nu} \quad \text{for all} \quad \nu, \mu \in P^+, \tag{4.1}$$

which is the second assertion of Proposition 2.3.

4.2. The next lemma proves (4.1) when $\operatorname{ht}\mu \le 1$.

Lemma. *Let $\nu \in P^+$ and $r \in [0, n]$ be the unique integer satisfying $\nu - \omega_r \in Q^+$. Then*

$$h^{\omega_k}_{\nu,0} = q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)}\delta_{k,r}, \quad k \in [0, n].$$

Proof. The lemma is proved by induction on ν with respect to the partial order on P^+. If $\nu = \omega_k$ for some $k \in [0, n]$ then $\omega_k - \omega_r \in Q^+$ if and only if $k = r$ and the result follows since $h^{\mu}_{\nu,0} = 0$ if $\nu - \mu \notin Q^+$ (see (2.6)). Assume the result for all $\nu' \in P^+$ with $\omega_r \le \nu' < \nu$, in particular we have $\operatorname{ht}\nu \ge 2$. Using (2.7) and the inductive hypothesis gives,

$$0 = \sum_{\mu \in P^+} g^{\mu}_{0,\nu} h^{\omega_k}_{\mu,0} = h^{\omega_k}_{\nu,0} + \sum_{\omega_k < \mu} g^{\mu}_{0,\nu} h^{\omega_k}_{\mu,0} = h^{\omega_k}_{\nu,0} + \delta_{k,r} \sum_{\mu < \nu} g^{\mu}_{0,\nu} q^{\frac{1}{2}(\mu+\omega_r,\ \mu-\omega_r)}. \tag{4.2}$$

Equation (3.2a) gives

$$q^{\frac{1}{2}(\nu+\omega_r,\nu-\omega_r)} + \sum_{\mu<\nu} q^{\frac{1}{2}(\mu+\omega_r,\mu-\omega_r)} g^{\mu}_{0,\nu} = 0,$$

and substituting in (4.2) proves the Lemma. □

4.3. We need several properties of the elements $h^{\mu}_{\nu,\lambda}$.

Lemma. *Suppose that $(\nu, \lambda) \in P^+ \times P^+$ is compatible or that $(\nu, \lambda) = (\omega_m, \lambda)$ with $\min \lambda = m$. Then*

$$h^{\mu}_{\nu,\lambda} = \sum_{\mu' \in P^+} q^{(\lambda+\nu-\mu',\nu)} g^{\mu'-\nu}_{0,\lambda} h^{\mu}_{\mu',0}.$$

Proof. Recall from Section 2 that

$$\operatorname{ch}_{\mathrm{gr}} M(\nu, \lambda) = \sum_{\mu' \in P^+} g^{\mu'}_{\nu,\lambda} \operatorname{ch}_{\mathrm{gr}} M(\mu', 0) = \sum_{\mu',\mu \in P^+} g^{\mu'}_{\nu,\lambda} h^{\mu}_{\mu',0} \operatorname{ch}_{\mathrm{gr}} M(0, \mu).$$

If (ν, λ) is compatible (resp. $(\nu, \lambda) = (\omega_m, \lambda + \omega_m)$) then using Proposition 3.2 (resp. Proposition 3.4) gives

$$\sum_{\mu \in P^+} h^{\mu}_{\nu,\lambda} \operatorname{ch}_{\mathrm{gr}} M(0, \mu) = \operatorname{ch}_{\mathrm{gr}} M(\nu, \lambda) = \sum_{\mu',\mu \in P^+} q^{(\lambda+\nu-\mu',\nu)} g^{\mu'-\nu}_{0,\lambda} h^{\mu}_{\mu',0} \operatorname{ch}_{\mathrm{gr}} M(0, \mu).$$

The lemma follows by equating coefficients of $\mathrm{ch}_{\mathrm{gr}}\, M(0,\mu)$. □

4.4. Suppose that (ν,λ) is compatible. Recall from Lemma 3.1 that,

$$h^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu-2\omega_j,\lambda+2\omega_j} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda}, \quad \text{if} \quad \nu(h_j)\geq 2, \tag{4.3}$$

and if $\nu \in P^+(1)$ with $\max\nu = m < \min\lambda_1 = p$ then

$$h^{\mu}_{\nu,\lambda} = h^{\mu}_{\nu-\omega_m,\lambda+\omega_m} + q^{\frac{1}{2}(\lambda,\alpha_{m,p})} h^{\mu}_{\nu-\omega_m+\omega_{m-1},\lambda-\omega_p+\omega_{p+1}}. \tag{4.4}$$

Lemma. *For (ν,λ) compatible and $k\in[1,n]$, we have*

$$h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k} = q^{(\lambda+\nu-\mu,\omega_k)} h^{\mu}_{\nu,\lambda},$$

and hence for $j\in[1,n]$ with $\nu(h_j)\geq 2$,

$$h^{\mu}_{\nu,\lambda} = q^{(\lambda+\nu-\mu,\omega_j)} h^{\mu-2\omega_j}_{\nu-2\omega_j,\lambda} + q^{(\nu+\lambda_0,\alpha_j)-1} h^{\mu}_{\nu-\alpha_j,\lambda}. \tag{4.5}$$

Proof. The proof is by induction on the partial order on compatible pairs (see Section 2.8). Induction obviously begins for the minimal elements $(0,\omega_i)$, $i\in[0,n]$ since $h^{\mu}_{0,\lambda} = \delta_{\lambda,\mu}$ for all $\lambda,\mu\in P^+$. Applying the inductive hypothesis to the right hand side of (4.3) we get if $\nu(h_j)\geq 2$ for some $j\in[1,n]$,

$$\begin{aligned} h^{\mu}_{\nu,\lambda} &= q^{-(\lambda+\nu-\mu,\omega_k)} \left(h^{\mu+2\omega_k}_{\nu-2\omega_j,\lambda+2\omega_j+2\omega_k} + q^{(\nu+\lambda_0+\omega_k,\alpha_j)-1} h^{\mu+2\omega_k}_{\nu-\alpha_j,\lambda+2\omega_k} \right) \\ &= q^{-(\lambda+\nu-\mu,\omega_k)} h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k}, \end{aligned}$$

where the second equality follows by using (4.3) again on the compatible pair $(\nu,\lambda+2\omega_k)$. If $\nu\in P^+(1)$ then the inductive hypothesis applied to the right hand side of (4.4) gives

$$\begin{aligned} h^{\mu}_{\nu,\lambda} &= q^{-(\lambda+\nu-\mu,\omega_k)} \left(h^{\mu+2\omega_k}_{\nu-\omega_m,\lambda+\omega_m+2\omega_k} + q^{(\lambda_0+\omega_k,\alpha_{m,p})+1} h^{\mu+2\omega_k}_{\nu-\omega_m+\omega_{m-1},\lambda+2\omega_k-\omega_p+\omega_{p+1}} \right) \\ &= q^{-(\lambda+\nu-\mu,\omega_k)} h^{\mu+2\omega_k}_{\nu,\lambda+2\omega_k}, \end{aligned}$$

where the last equality is a further application of (4.4) to the compatible pair $(\nu,\lambda+2\omega_k)$. Equation (4.5) is now immediate from (4.3) and the proof of the Lemma is complete. □

4.5.

Lemma. *Let $m\in[1,n]$ and $\lambda = 2\lambda_0+\lambda_1\in P^+$ with $m<\min\lambda_1$ if $\lambda_0\neq 0$ and $m\leq\min\lambda_1$ if $\lambda_0 = 0$. If $\mu\in P^+$ with $\mu(h_m)\in 2\mathbb{Z}_+ + 1$ and $\lambda+\omega_n\neq\mu$, then $h^{\mu}_{\omega_m,\lambda} = 0$.*

Proof. If $m < \min \lambda_1$ we claim that a stronger statement is true; namely

$$h^{\mu}_{\omega_m,\lambda} \neq 0 \implies \mu = \lambda + \omega_m \quad \text{or} \quad \mu(h_s) \in 2\mathbb{Z}_+ \ \text{ for all } s \in [m, p].$$

We prove the claim by an induction on m with induction beginning at $m = 0$ since $h^{\mu}_{0,\lambda} = \delta_{\lambda,\mu}$. If $m \geq 1$ then using (4.4) we have,

$$h^{\mu}_{\omega_m,\lambda} = 0 \implies \mu = \lambda + \omega_m \quad \text{or} \quad h^{\mu}_{\omega_{m-1},\lambda-\omega_p+\omega_{p+1}} \neq 0.$$

The inductive hypothesis applies to the pair $(\omega_{m-1}, \lambda - \omega_p + \omega_{p+1})$ and so

$$h^{\mu}_{\omega_{m-1},\lambda-\omega_p+\omega_{p+1}} \neq 0 \implies \mu = \lambda + \omega_m - \alpha_{m,p} \quad \text{or} \quad \mu(h_s) \in 2\mathbb{Z}_+ \text{ for all } \ s \in [m-1, p+1].$$

Since $(\lambda_1 + \omega_m - \alpha_{m,p})(h_s) = 0$ for all $s \in [m, p]$ the inductive step follows and the claim is proved.

It remains to prove the lemma when $\lambda \in P^+(1)$ and $m = \min \lambda_1$. Then Lemma 3.1 gives

$$h^{\mu}_{\omega_m,\lambda} \neq 0 \implies \mu = \lambda + \omega_m \text{ or } h^{\mu}_{\omega_{m-1},\lambda-\omega_m+\omega_{m+1}} \neq 0.$$

If $\lambda - \omega_m = \omega_{m+1}$ then the isomorphism in (2.4) gives

$$h^{\mu}_{\omega_{m-1},2\omega_{m+1}} \neq 0 \iff \mu = \omega_{m-1} + 2\omega_{m+1},$$

and the result follows in this case. Otherwise $\lambda - \omega_m \neq \omega_{m+1}$ and the result has been proved for $(\omega_{m-1}, \lambda - \omega_m + \omega_{m+1})$. Hence

$$h^{\mu}_{\omega_{m-1},\lambda-\omega_m+\omega_{m+1}} \neq 0 \implies \mu(h_m) \in 2\mathbb{Z}_+ \quad \text{or} \quad \mu = \lambda - \omega_m + \omega_{m+1} + \omega_{m-1}.$$

Since $(\lambda - \omega_m + \omega_{m+1} + \omega_{m-1})(h_m) = 0$ the proof of the Lemma is complete. □

4.6. Proof of equation (4.1)

The equality obviously holds if $\nu - \mu \notin Q^+$. If $\nu - \mu = \sum_{i=1}^n k_i\alpha_i \in Q^+$ we set $\mathrm{ht}_r(\nu - \mu) = \sum_{i=1}^n k_i$ and proceed by induction on $\mathrm{ht}_r(\nu - \mu)$. The induction begins when $\mathrm{ht}_r(\nu - \mu) = 0$ since then $\nu = \mu$ and $h^{\nu}_{\nu,0} = 1$. Assume the result holds for all pairs (ν, μ) with $\mathrm{ht}_r(\nu - \mu) < N$.

We prove the result for $\mathrm{ht}_r(\nu - \mu) = N$ by a further induction on $\mathrm{ht}\,\mu$. Lemma 4.2 shows that this second induction begins when $\mathrm{ht}\,\mu \leq 1$. Assume the result holds for all ν with $\mathrm{ht}_r(\nu - \mu) \leq N$ and $1 \leq \mathrm{ht}\,\mu < s$. If $\nu(h_j) \geq 2$ for some $j \in [1, n]$ then the inductive hypothesis on $\mathrm{ht}\,\mu$ (resp. $\mathrm{ht}_r(\nu - \mu)$) applies to the first (resp. second term) on the right hand sides of (4.5) with $(\lambda = 0)$. Hence

$$h^{\mu}_{\nu,0} = q^{(\omega_j,\nu-\mu)} p^{\mu-2\omega_j}_{\nu-2\omega_j} + q^{(\nu,\alpha_j)-1} p^{\mu}_{\nu-\alpha_j} = p^{\mu}_{\nu}.$$

In particular the inductive step has been proved for $\operatorname{ht}\mu \in \{s, s+1\}$ provided that there exists $j \in [1,n]$ with $\nu(h_j) \geq 2$.

If $\nu \in P^+(1)$ we let $m = \min\nu$ and consider two cases. If $\mu(h_m) \in 2\mathbb{Z}_+ + 1$ we use Lemma 4.5, followed by Lemma 4.3 applied to the compatible pair $(\omega_m, \nu - \omega_m)$, to get

$$0 = h^{\mu}_{\omega_m,\nu-\omega_m} = \sum_{\mu' \leq \nu-\omega_m} q^{(\nu-\omega_m-\mu',\omega_m)} g^{\mu'}_{0,\nu-\omega_m} h^{\mu}_{\mu'+\omega_m,0}. \tag{4.6}$$

If $\mu' < \nu - \omega_m$ then $\operatorname{ht}_r(\mu' + \omega_m - \mu) < \operatorname{ht}_r(\nu - \mu)$ and so the inductive hypothesis gives

$$h^{\mu}_{\mu'+\omega_m,0} = p^{\mu}_{\mu'+\omega_m} = q^{(\mu'-\mu+\omega_m,\omega_m)} p^{\mu-\omega_m}_{\mu'} = q^{(\mu'-\mu+\omega_m,\omega_m)} h^{\mu-\omega_m}_{\mu',0}.$$

Substituting in (4.6), we have

$$0 = h^{\mu}_{\nu,0} + q^{(\nu-\mu,\omega_m)} \sum_{\mu' < \nu-\omega_m} g^{\mu'}_{0,\nu-\omega_m} h^{\mu-\omega_m}_{\mu',0}.$$

Equation (2.7) and the inductive hypothesis applied to $\mu - \omega_m$ now give

$$h^{\mu}_{\nu,0} = q^{(\nu-\mu,\omega_m)} h^{\mu-\omega_m}_{\nu-\omega_m,0} = q^{(\nu-\mu,\omega_m)} p^{\mu-\omega_m}_{\nu-\omega_m} = p^{\mu}_{\nu}$$

as needed.

It remains to consider the case when $\mu(h_m) \in 2\mathbb{Z}_+$ and $\nu \neq \mu$. Since $(\nu+\omega_m)(h_m) = 2$, it follows from the first part of the proof that

$$h^{\mu+\omega_m}_{\nu+\omega_m,0} = p^{\mu+\omega_m}_{\nu+\omega_m} = q^{(\nu-\mu,\omega_m)} p^{\mu}_{\nu}. \tag{4.7}$$

Using Lemma 4.5 and Lemma 4.3 for the pair (ω_m, ν) gives

$$0 = h^{\mu+\omega_m}_{\omega_m,\nu} = \sum_{\mu' \leq \nu+\omega_m} q^{(\nu+\omega_m-\mu',\omega_m)} g^{\mu'-\omega_m}_{0,\nu} h^{\mu+\omega_m,0}_{\mu'}. \tag{4.8}$$

Since the inductive hypothesis applies to $h^{\mu+\omega_m}_{\mu',0}$ if $\mu' < \nu + \omega_m$ we have

$$h^{\mu+\omega_m}_{\mu',0} = p^{\mu+\omega_m}_{\mu'} = q^{(\mu'-\mu-\omega_m,\omega_m)} p^{\mu}_{\mu'-\omega_m} = q^{(\mu'-\mu-\omega_m,\omega_m)} h^{\mu}_{\mu'-\omega_m,0}.$$

Substituting in (4.8) and using (2.7) we get

$$0 = h^{\mu+\omega_m}_{\lambda+\omega_m} + q^{(\lambda-\mu,\omega_m)} \sum_{\mu'<\lambda} g^{\mu'}_{0,\lambda} h^{\mu}_{\mu'} = h^{\mu+\omega_m}_{\lambda+\omega_m} - q^{(\lambda-\mu,\omega_m)} h^{\mu}_{\lambda}.$$

An application of equation (4.7) completes the proof.

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